Forcings with the countable chain condition and Parametrized $\diamondsuit$ principles

Hiroaki Minami
Graduate School of Science and Technology,
Kobe University

Introduction 1

- Jensen introduced $\diamondsuit$ principle.

  $\diamondsuit \quad \exists\langle A_\alpha \subset \alpha : \alpha < \omega_1\rangle$ such that $\forall X \subset \omega_1$ the set $\{\alpha < \omega_1 : X \cap \alpha = A_\alpha\}$ is stationary.

- Devlin and Shelah introduced weak diamond principle.

  $\forall F : 2^{<\omega_1} \to 2 \ \exists g : \omega_1 \to 2$ such that $\forall f : \omega_1 \to 2$ the set $\{\alpha \in \omega_1 : F(f \restriction \alpha) = g(\alpha)\}$ is stationary.

- Hrušák introduced $\diamondsuit_{\mathfrak{d}}$.

  $\diamondsuit_{\mathfrak{d}} \quad \exists\langle g_\alpha : \omega \le \alpha < \omega_1\rangle$ such that g_α is a function from α to ω and $\forall f : \omega_1 \to \omega$ $\exists \alpha \ge \omega$ with $f \restriction \alpha \le^* g_\alpha$.

- Moore, Hrušák and Džamonja introduced Parametrized $\diamondsuit$ principles.

Introduction 2

- By using ω_2-stage countable support iteration of proper forcing notions Moore, Hrušák and Džamonja construct some models of Parametrized $\diamondsuit$ principles .

- By using ω_1-stage finite support iteration of c.c.c forcing notions some models of Parametrized $\diamondsuit$ principles are given.

Question 1. *Can we build models of parametrized* $\diamondsuit$ *principles by* ω_2*-stage finite support iteration of c.c.c forcing notions?*

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1 Definition of Parametrized $\diamondsuit$ principles

Borel invariant

Definition 2 (Vojtáš)(Moore, Hrušák, Džamonja)**.**
(A, B, E) *is an invariant if*

(1) $|A|$, $|B| \leq \mathfrak{c}$,

(2) $E \subset A \times B$,

(3) $\forall a \in A \exists b \in B(\ (a, b) \in E)$,

(4) $\forall b \in B \exists a \in A(\ (a, b) \notin E)$.

We will write aEb *instead of* $(a, b) \in E$.

Definition 3. *(Blass)*
An invariant (A, B, E) *is* **Borel** *if* A, B *and* E *are Borel subsets of some Polish space.*

Definition 4.
If (A, B, E) is an invariant, then its evaluation $\langle A, B, E\rangle$ is given by

$$\langle A, B, E\rangle = \text{min}\{|X| : X \subset B, \forall a \in A \exists b \in X (aEb)\}.$$

Example
Let $\mathcal{M}$ be a meager ideal and $\mathcal{N}$ be a null ideal. Then

$$\begin{array}{cccc} (\mathbb{R}, \mathcal{N}, \in) & (\mathcal{M}, \mathbb{R}, \not\ni) & (\mathcal{M}, \mathcal{M}, \subset) & (\mathcal{N}, \mathcal{N}, \subset) \\ & (\omega^\omega, \omega^\omega, \not\geq^*) & (\omega^\omega, \omega^\omega, \leq^*) & \\ (\mathcal{N}, \mathcal{N}, \not\supset) & (\mathcal{M}, \mathcal{M}, \not\supset) & (\mathbb{R}, \mathcal{M}, \in) & (\mathcal{N}, \mathbb{R}, \not\ni) \end{array}$$

are all Borel invariants.

$$\langle \mathbb{R}, \mathcal{M}, \in\rangle = \text{cov}(\mathcal{M}).$$

$$\langle \mathcal{N}, \mathbb{R}, \not\ni\rangle = \text{non}(\mathcal{N}).$$

$\boxed{\text{Parametrized } \diamondsuit \text{ principles}}$

Definition 5. *Let* A *be a Borel set in some Polish Space. Then* $F : 2^{<\omega_1} \to A$ *is Borel if* $\forall \alpha < \omega_1 (F \restriction 2^\alpha$ *is a Borel function).*

Definition 6. (A, B, E)*: Borel invariant* $\diamondsuit(A, B, E) \equiv \forall$ *Borel function* $F : 2^{<\omega_1} \to A$ $\exists g : \omega_1 \to B$ *such that* $\forall f : \omega_1 \to 2$ *the set*

$$\{\alpha < \omega_1 : F(f \restriction \alpha) E g(\alpha)\}$$

is stationary.
The witness g *for* F *in this statement will be called* $\diamondsuit(A, B, E)$*-sequence for* F*.*

$\diamondsuit(A, B, E)$ and $\langle A, B, E\rangle$

Proposition 7. Suppose (A, B, E) is a Borel invariant and $\diamondsuit(A, B, E)$ holds. Then $\langle A, B, E\rangle \leq \omega_1$ holds.

$\diamondsuit$ and $\diamondsuit(A, B, E)$

Proposition 8. (A, B, E)*: Borel invariant.*
$\diamondsuit$ *implies* $\diamondsuit(A, B, E)$*.*

Borel-Tukey order

Definition 9. (Borel Tukey ordering)
$(A_1, B_1, E_1) \leq_T^B (A_2, B_2, E_2)$ if there exist Borel maps ϕ, ψ such that

- $\phi : A_1 \to A_2$
- $\psi : B_2 \to B_1$
- $\forall a \in A_1 \forall b \in B_2 ($ $\phi(a) E_2 b$ implies $a E_1 \psi(b)$ $)$.

Example

Cichoń's diagram

$$\begin{tikzcd}
(\mathbb{R}, \mathcal{N}, \in) \arrow[dd] & (\mathcal{M}, \mathbb{R}, \not\ni) \arrow[l] \arrow[d] & (\mathcal{M}, \mathcal{M}, \subset) \arrow[l] \arrow[d] & (\mathcal{N}, \mathcal{N}, \subset) \arrow[l] \arrow[dd] \\
 & (\omega^\omega, \omega^\omega, \not\geq^*) \arrow[d] & (\omega^\omega, \omega^\omega, \leq^*) \arrow[l] \arrow[d] & \\
(\mathcal{N}, \mathcal{N}, \not\supset) & (\mathcal{M}, \mathcal{M}, \not\supset) \arrow[l] & (\mathbb{R}, \mathcal{M}, \in) \arrow[l] & (\mathcal{N}, \mathbb{R}, \not\ni) \arrow[l]
\end{tikzcd}$$

The direction of the arrow is from larger to smaller in the Borel Tukey order.

Borel-Tukey order and Parametrized $\diamondsuit$ principles

Proposition 10. Suppose $\diamondsuit(A_2, B_2, E_2)$ and $(A_1, B_1, E_1) \leq^B_T (A_2, B_2, E_2)$ holds.
Then $\diamondsuit(A_1, B_1, E_1)$ holds.

2. Construction of diamonds

Theorem 11. Let $\mathbb{P}_{\omega_1}$ be an ω_1-stage finite support iteration of c.c.c forcing notions such that for all $\alpha \in \omega_1$

$$\exists b \in B \cap V^{\mathbb{P}_{\omega_1}} \forall a \in A \cap V^{\mathbb{P}_\alpha}(aEb)$$

Then $V^{\mathbb{P}_{\omega_1}} \models \diamondsuit(A, B, E)$.

Proof. Let $F : 2^{<\omega_1} \to A$ be a Borel function in $V^{\mathbb{P}_{\omega_1}}$. For each $\alpha \in \omega_1$, let $r_\alpha \in V^{\mathbb{P}_{\omega_1}}$ be a real coding $F \restriction 2^\alpha$. Then define $h : \omega_1 \to \omega_1$ strictly increasing such that $r_\alpha \in V^{\mathbb{P}_{h(\alpha)}}$.
Then define $g : \omega_1 \to B$ so that for each $\alpha \in \omega_1$

$$\forall a \in A \cap V^{\mathbb{P}_{h(\alpha)}}(\ aEg(\alpha)\ ).$$

Claim 11.1. g is $\diamondsuit(A, B, E)$-sequence for F.

Let $f : \omega_1 \to 2$. Then define

$$C_f = \{\alpha \in \omega_1 : f \restriction \alpha \in V^{\mathbb{P}_\alpha}\}.$$

Since $\mathbb{P}_{\omega_1}$ is c.c.c, C_f is club. Then if $\alpha \in C_f$, then $F(f \restriction \alpha) \in A \cap V^{\mathbb{P}_{h(\alpha)}}$. So $F(f \restriction \alpha)Eg(\alpha)$. Hence g is a $\diamondsuit(A, B, E)$-sequence for F.

Claim$\blacksquare$ Theorem $\square$

$\boxed{\omega_2\text{-stage countable support iteration}}$

Theorem 12. *(Moore, Hrušák and Džamonja) Suppose that* $\langle \mathcal{Q}_\alpha : \alpha < \omega_2 \rangle$ *is a sequence of Borel partial orders such that for each* $\alpha < \omega_2$ $\mathcal{Q}_\alpha$ *is equivalent to* $\wp(2)^+ \times \mathcal{Q}_\alpha$ *as a forcing notion and let* $\mathcal{P}_{\omega_2}$ *be the countable support iteration of this sequence.*

If $\mathcal{P}_{\omega_2}$ *is proper and* (A, B, E) *is a Borel invariant then* $\mathcal{P}_{\omega_2}$ *forces* $\langle A, B, E \rangle \leq \omega_1$ *iff* $\mathcal{P}_{\omega_2}$ *forces* $\diamondsuit(A, B, E)$*.*

2. Suslin forcing and (A, B, E)-niceness

Main idea. Let (A, B, E) be a Borel invariant. Let $\mathbb{P}_{\omega_2}$ be the finite support iteration of c.c.c forcing notions. Let $F : 2^{<\omega_1} \to A$ be a Borel function in $V^{\mathbb{P}_{\omega_2}}$. Without loss of generality, we can assume $F \in V$. Let $\dot{f} : \omega_1 \to 2$ in $V^{\mathbb{P}_{\omega_2}}$. Then the set

$$C_{\dot{f}} = \{\alpha \in \omega_1 : f \restriction \alpha \in V^{\mathbb{P}_\alpha * \mathbb{P}_{[\omega_1, \omega_2)}}\}$$

contains club. Let $c_\alpha \in V^{\mathbb{P}_{\omega_1}}$ be a Cohen real over $V^{\mathbb{P}_\alpha}$. Work in $V^{\mathbb{P}_\alpha}$, if for any $\mathbb{P}_{[\omega_1, \omega_2)}$-name $\dot{x}$ such that $\Vdash_{\mathbb{P}_{[\omega_1, \omega_2)}} \dot{x} \in A$

$$\Vdash_{\mathbb{P}_{[\alpha, \omega_1)} * \dot{\mathbb{P}}_{[\omega_1, \omega_2)}} \dot{x} E \dot{c}_\alpha$$

Then $\langle c_\alpha : \alpha \in \omega_1 \rangle$ is a $\Diamond(A, B, E)$-sequence for F.

Question 13. *Does this proof work ?*

We should check following

(i) If $\dot{x}$ be a $\mathbb{P}_{\omega_2}$-name for a real, then can we assume $\dot{x}$ be a $\mathbb{P}_\alpha * \dot{\mathbb{P}}_{[\omega_1,\omega_2)}$-name for some $\alpha < \omega_1$?

(ii) Let $\dot{x}$ be a $\mathbb{P}_\alpha * \dot{\mathbb{P}}_{[\omega_1,\omega_2)}$-name such that $\Vdash_{\mathbb{P}_\alpha * \dot{\mathbb{P}}_{[\omega_1,\omega_2)}} \dot{x} \in A$. Let $\dot{r}$ be a $\mathbb{P}_{\omega_1}$-name such that for any $x \in A \cap V^{\mathbb{P}_\alpha}$

$$\Vdash_{\mathbb{P}_{\omega_1}} xE\dot{r}.$$

Can we assume $\Vdash_{\mathbb{P}_{\omega_1} * \dot{\mathbb{P}}_{[\omega_1,\omega_2)}}$ "$\dot{x}E\dot{r}$" ?

Suslin forcing

Definition 14. *A forcing notion* $\mathbb{P} = \langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ *has a Suslin definition if* $\mathbb{P} \subset \omega^\omega$, $\leq_{\mathbb{P}} \subset \omega^\omega \times \omega^\omega$ *and* $\perp_{\mathbb{P}} \subset \omega^\omega \times \omega^\omega$ *are* Σ^1_1.

$\mathbb{P}$ *is Suslin if* $\mathbb{P}$ *is c.c.c and has a Suslin definition.*

For convenience we will interpret Suslin forcing notion in forcing extensions using its code rather than taking the ground model forcing notion.

Lemma 15. *Let $\mathbb{P}_\kappa$ be the κ-stage finite support iteration of Suslin forcing notions. Let $\dot{x}$ be a $\mathbb{P}_\kappa$-name for a real. Then there exists countable $I \subset \kappa$ such that $\dot{x}$ is $\mathbb{P}_I$-name.*

Hence if $\dot{f} : \omega_1 \to 2$ is a $\mathbb{P}_{\omega_2}$-name, then the set

$$C_{\dot{f}} = \{\alpha < \omega_1 : \dot{f} \restriction \alpha \text{ is } \mathbb{P}_\alpha * \dot{\mathbb{P}}_{[\omega_1,\omega_2)}\text{-name}\}$$

contains club.

(A,B,E) with$\langle E_n : n \in \omega\rangle$ and $\langle U^n : n \in \omega\rangle$

We will deal with Borel invariants (A,B,E) with $\langle E_n : n \in \omega\rangle$ and $\langle U^n : n \in \omega\rangle$ satisfying the following:

(0) E_n is a Borel set for $n \in \omega$,

(1) $E_n \subset A \times B$ and $E = \bigcap_{n\in\omega} E_n$,

(2) $E_{n+1} \subset E_n$,

(3) $U^n : A \to \wp(A)$ such that $U^n(x)$ is a Borel set,

(4) xE_ny implies that there exists $m \geq n$ such that $U^m(x) \subset \{z \in A : zE_ny\}$,

(5) $U^m(x) \subset \{z \in A : zE_ny\}$ is absolute with parameters x, y, U^m and E_n and

(6) If $x \in U^n(y)$, then $U^n(x) \subset U^n(y)$.

Example

For $(\omega^\omega, \not\geq^*)$ let

$$x \not\geq_n y \text{ if } \exists m \geq n(x(m) < y(n))$$

and $U^n(x) = \bigcup_{m \leq x(n)} [\langle n, m \rangle]$.

Then $\langle \not\geq_n : n \in \omega \rangle$ and $\langle U^n : n \in \omega \rangle$ satisfy (0)-(6).

$\boxed{\text{(A,B,E)-nice}}$

Definition 16. *Let* (A, B, E) *be a Borel invariant with* $\langle E_n : n \in \omega \rangle$ *and* $\langle U^n : n \in \omega \rangle$ *satisfying (0)-(6). Let* $\mathbb{P}$ *be a forcing notion and* $\mathcal{Q}$ *be a Suslin forcing notion or finite support iteration of Suslin forcing notions.*

Then $\mathcal{Q}$ *is* (A, B, E)*-nice with* $\langle E_n : n \in \omega \rangle$ *and* $\langle U^n : n \in \omega \rangle$ *for* $\mathbb{P}$ *if* $\forall \dot{x}$ $\mathcal{Q}$*-names such that* $\Vdash_{\mathcal{Q}}$ *"*$\dot{x} \in A$*",* $\forall (p, \dot{q}) \in \mathbb{P} * \dot{\mathcal{Q}}$ $\exists x \in A \cap V$ $\exists m \in \omega$ *such that* $\forall r \leq_{\mathbb{P}} p$ $\forall n \geq m$ $\exists q\prime \in \mathcal{Q}$ *such that* $(1, q\prime) \| (r, \dot{q})$ *and* $q\prime \Vdash_{\mathcal{Q}} \dot{x} \in U^n(x)$*.*

Definition 17. *Let μ be the standard product measure on 2^ω, $\mathcal{B}$ the family of all Borel sets on 2^ω and $\mathcal{N}_\omega = \{X \in \mathcal{B} : \mu(X) = 0\}$. Then define $\mathbb{B} = \mathcal{B}/\mathcal{N}_\omega$. It is ordered by*

$$A \leq_{\mathbb{B}} B \text{ if } \mu(A \setminus B) = 0.$$

Definition 18. *The Hechler forcing notion is defined as follows:*

$$\langle s, f\rangle \in \mathbb{D} \text{ if } s \in \omega^{<\omega},\ f \in \omega^\omega \text{ and } s \subset f.$$

It is ordered by

$$\langle s, f\rangle \leq \langle t, g\rangle \text{ if } s \supset t \text{ and } g \leq f.$$

Definition 19. *The eventually different forcing notion is defined as follows:*

$$\langle s, H\rangle \in \mathbb{E} \text{ if } s \in \omega^{<\omega} \text{ and } H \in [\omega^\omega]^{<\omega}$$

It is ordered by $\langle s, H\rangle \leq \langle s', H'\rangle$ if $s \supset s', H \supset H'$ and

$$\text{for all } f \in H' \text{ for all } j \in [\left|s'\right|, |s|)\ s(j) \neq f(j).$$

Example of (A,B,E)-niceness

Proposition 20. *Suppose I is countable subset of some ordinal κ. Then*

(1) $\mathbb{D}_I$ is $(2^\omega, 2^\omega, =^{\exists\infty}_{I_n})$-nice with $\langle =^{m\geq n}_{I_m} : n \in \omega\rangle$ and $\langle [* \restriction I_n] : n \in \omega\rangle$ for $\mathbb{D}_{\omega_1}$

(2) $\mathbb{B}_I$ is $(\omega^\omega, \not\geq^*)$-nice with $\langle \not\geq_n : n \in \omega\rangle$ and $\langle \bigcup_{m \leq *(n)} [\langle n, m\rangle] : n \in \omega\rangle$ for $\mathbb{B}_{\omega_1}$.

(3) $\mathbb{E}_I$ is $(2^\omega, 2^\omega, =^{\exists\infty}_{I_n})$-nice with $\langle =^{m\geq n}_{I_m} : n \in \omega\rangle$ and $\langle [* \restriction I_n] : n \in \omega\rangle$ for $\mathbb{E}_\omega$ and $(\omega^\omega, \not\geq^*)$-nice with $\langle \not\geq_n : n \in \omega\rangle$ and $\langle \bigcup_{m \leq *(n)} [\langle n, m\rangle] : n \in \omega\rangle$ for $\mathbb{E}_{\omega_1}$.

(4) $(\mathbb{B} * \dot{\mathbb{D}})_I$ is $(\mathbb{LOC}, \omega^\omega, \not\ni)$-nice with $\langle \not\ni_n : n \in \omega\rangle$ and $\langle \bigcup_{s \subset *(n)} [\langle n, s\rangle] : n \in \omega\rangle$ for $(\mathbb{B} * \dot{\mathbb{D}})_{\omega_1}$.

Theorem 21. *[Minami]* (A, B, E) *Borel invariant with* $\langle E_n : n \in \omega\rangle$ *and* $\langle U^n : n \in \omega\rangle$ *satisfying (0)-(6). Let* $\mathbb{P}$ *be a forcing notion such that*

$\exists \dot{r}$ $\mathbb{P}$*-name* ($\Vdash_{\mathbb{P}}$ *"*$\dot{r} \in B$ *and* $xE\dot{r}''$ *for* $x \in A \cap V$)

and let $\mathcal{Q}$ *be a Suslin forcing notion or the finite support iteration of Suslin forcing notions. If* $\mathcal{Q}$ *is* (A, B, E)*-nice for* $\mathbb{P}$ *and* $\dot{x}$ *is a* $\mathcal{Q}$*-name for an element of* $A \cap V^{\mathbb{P}}$*, then* $\Vdash_{\mathbb{P} * \dot{\mathcal{Q}}} \dot{x}E\dot{r}$*.*

Theorem 22. *Let (A, B, E) be a Borel invariant with $\langle E_n, n \in \omega \rangle$ and $\langle U^n : n \in \omega \rangle$ satisfying (0)-(6). Let $\mathbb{P}_{\omega_2}$ be a ω_2-stage finite support iteration of Suslin forcing notion and*

(a) *for all $\beta < \omega_2$ there exists a $\mathbb{P}_{\beta+\omega_1}$-name $\dot{r}$ for an element of A such that $\Vdash_{\mathbb{P}_{\beta+\omega_1}}$ "$x E \dot{r}$" for $x \in A \cap V^{\mathbb{P}_\beta}$.*

(b) *for all $\beta < \omega_2$ for all I countable subset of $\omega_2 \setminus (\beta + \omega_1)$*
$V^{\mathbb{P}_\beta} \models$ " $\mathbb{P}_I$ is $(A, B, E)-$nice for $\mathbb{P}_{[\beta,\beta+\omega_1)}$".

Then $\mathbb{P}_{\omega_2} \models \diamondsuit(A, B, E)$.

Corollary 23. *Each of the following are relatively consistent with ZFC:*

(i) $\mathfrak{c} = add(\mathcal{M}) = \omega_2 + \diamondsuit(cov(\mathcal{N}))$ *(see Diagram 1).*

(ii) $\mathfrak{c} = cov(\mathcal{N}) = cov(\mathcal{M}) = \omega_2 + \diamondsuit(\mathfrak{b})$ *(see Diagram 2).*

(iii) $\mathfrak{c} = non(\mathcal{M}) = cov(\mathcal{M}) = \omega_2 + \diamondsuit(\mathfrak{b}) + \diamondsuit(cov(\mathcal{N}))$ *(see Diagram 3).*

(iv) $\mathfrak{c} = cov(\mathcal{N}) = add(\mathcal{M}) = \omega_2 + \diamondsuit(add(\mathcal{N}))$ *(see Diagram 4).*

Proof. (i) Suppose $V \models \mathrm{CH}$. Then

$$V^{\mathbb{D}_{\omega_2}} \models \Diamond(\mathrm{cov}(\mathcal{N})) + \mathfrak{c} = \mathrm{add}(\mathcal{M}) = \omega_2.$$

$$\begin{tikzcd}
 & & & & \omega_2 \\
\Diamond(\mathrm{cov}(\mathcal{N})) \arrow[r, dash] \arrow[dd, dash] & \omega_2 \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[ur, dash] \arrow[dd, dash] & \\
 & \omega_2 \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[d, dash] & & \\
\Diamond(\mathrm{add}(\mathcal{N})) \arrow[r, dash] & \omega_2 \arrow[r, dash] & \omega_2 \arrow[r, dash] & \omega_2 &
\end{tikzcd}$$

Diagram 1.

Cichoń's diagram for parametrized diamond looks as follows where a ω_2 means the corresponding evaluation of Borel invariant is ω_2 while parametrized diamonds principle for the others hold.

(ii) Suppose $V \models$ CH.
Then

$$V^{\mathbb{B}_{\omega_2}} \models \Diamond(\mathfrak{b}) + \mathfrak{c} = \mathrm{cov}(\mathcal{N}) = \mathrm{cov}(\mathcal{M}) = \omega_2.$$

$$\begin{tikzcd}
& & & & \omega_2 \\
\omega_2 \arrow[r, dash] \arrow[dd, dash] & \omega_2 \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[ur, dash] \arrow[dd, dash] & \\
& \Diamond(\mathfrak{b}) \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[d, dash] & & \\
\Diamond(\mathrm{add}(\mathcal{N})) \arrow[r, dash] & \Diamond(\mathrm{add}(\mathcal{M})) \arrow[r, dash] & \omega_2 \arrow[r, dash] & \omega_2 &
\end{tikzcd}$$

Diagram 2.

(iii) Suppose $V \models \mathrm{CH}$.
Then

$$\begin{aligned} V^{\mathbb{E}_{\omega_2}} \models {} & \diamondsuit(\mathrm{cov}(\mathcal{N})) + \diamondsuit(\mathrm{cov}(\mathcal{M})) + \\ & \mathfrak{c} = \mathrm{non}(\mathcal{M}) + \mathrm{cov}(\mathcal{M}) = \omega_2. \end{aligned}$$

$$\begin{tikzcd}
& & & & \omega_2 \\
\diamondsuit(\mathrm{cov}(\mathcal{N})) \arrow[r, dash] \arrow[dd, dash] & \omega_2 \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[ur, dash] \arrow[dd, dash] & \\
& \diamondsuit(\mathfrak{b}) \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[d, dash] & & \\
\diamondsuit(\mathrm{add}(\mathcal{N})) \arrow[r, dash] & \diamondsuit(\mathrm{add}(\mathcal{M})) \arrow[r, dash] & \omega_2 \arrow[r, dash] & \omega_2 &
\end{tikzcd}$$

Diagram 3.

(iv) Suppose $V \models$ CH.
Then

$$V^{(\mathbb{B}*\dot{\mathbb{D}})_{\omega_2}} \models \Diamond(\mathrm{add}(\mathcal{N}))+$$
$$\mathfrak{c} = \mathrm{cov}(\mathcal{N}) = \mathrm{add}(\mathcal{M}) = \omega_2.$$

$$\begin{tikzcd}
 & & & & \omega_2 \\
\omega_2 \arrow[r, dash] \arrow[dd, dash] & \omega_2 \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[ur, dash] \arrow[dd, dash] & \\
 & \omega_2 \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[d, dash] & & \\
\Diamond(\mathrm{add}(\mathcal{N})) \arrow[r, dash] & \omega_2 \arrow[r, dash] & \omega_2 \arrow[r, dash] & \omega_2 &
\end{tikzcd}$$

Diagram 4

$\square$