On the uniformity number of F_σ-ideals

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joint work with
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pair-splitting number and pair-reapin number

Theorem (Minami)

1. It is consistent that $\mathfrak{s}_{pair} > \mathfrak{d}$.
2. It is cosistent taht $\mathfrak{r}_{pair} < \mathfrak{b}$.

Corollary

1. $\mathfrak{r}_{pair} \leq \mathfrak{l}$.
2. $\mathfrak{s}_{pair} \geq$ trans-add($\mathcal{N}$).

The uniformity number and covering number of $\mathcal{G}_{FC}$

The finite chromatic ideal on $\omega \times \omega$ is defined by

$$\mathcal{G}_{FC} = \{A \subset [\omega]^2 : \chi(\omega, A) < \infty\}$$

where $\chi(\omega, A) = \min\{n \in \omega : \exists f : \omega \to n \wedge \forall a \in A\, |f[a]| = 2\}$.

Proposition

$\mathcal{G}_{FC}$ is an F_σ-ideal on $\omega \times \omega$.

Theorem (Hrušák and Meza-Alcántara)

1. $\mathrm{non}^*(\mathcal{G}_{FC}) \geq \mathfrak{r}_{pair}$.
2. $\mathrm{cov}^*(\mathcal{G}_{FC}) = \mathfrak{s}_{pair}$.

The uniformity number and the covering number of F_σ-ideals on ω

Question

For every Borel ideal $\mathcal{I}$ on ω,

1. $Con(\mathrm{cov}^*(\mathcal{I}) > \mathfrak{d})$?
2. $Con(\mathrm{non}^*(\mathcal{I}) < \mathfrak{b})$?
3. $\mathrm{non}^*(\mathcal{I}) \leq \mathfrak{l}$?
4. $\mathrm{cov}^*(\mathcal{I}) \geq \text{trans-add}(\mathcal{N})$?

Answer

Yes,

for every F_σ-ideal $\mathcal{I}$ on ω.

The uniformity number and the covering number of ideals on ω

Definition

$\mathcal{I} \subset \mathcal{P}(\omega)$ is an ideal on ω if

1. $\forall X, Y \in \mathcal{I}(X \cup Y \in \mathcal{I})$,
2. $\forall X, Y(X \subset Y \wedge Y \in \mathcal{I} \rightarrow X \in \mathcal{I})$,
3. $\omega \notin \mathcal{I}$ and
4. $Fin \subset \mathcal{I}$.

We call an ideal $\mathcal{I}$ tall if $\forall X \in [\omega]^\omega \exists I \in \mathcal{I}(|X \cap I| = \aleph_0)$.

Let $\mathcal{I}$ be a tall ideal.

$$\begin{aligned} \text{cov}^*(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{I} \wedge \forall X \in [\omega]^\omega \exists A \in \mathcal{A}(|A \cap X| = \aleph_0)\}. \\ \text{non}^*(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subset [\omega]^\omega \wedge \forall I \in \mathcal{I} \exists A \in \mathcal{A}(|A \cap I| < \aleph_0)\}. \end{aligned}$$

Katětov order

Definition

Let $\mathcal{I}$ and $\mathcal{J}$ be an ideal on ω.

$$\mathcal{I} \leq_K \mathcal{J} \quad \text{if} \quad \exists f : \omega \to \omega \forall I \in \mathcal{I}(f^{-1}[I] \in \mathcal{J}).$$

We call this order $\leq_K$ Katětov order.

Proposition

If $\mathcal{I} \leq_K \mathcal{J}$, then $\text{non}^*(\mathcal{I}) \leq \text{non}^*(\mathcal{J})$ and $\text{cov}^*(\mathcal{I}) \geq \text{cov}^*(\mathcal{J})$.

Dichotomy theorem

The eventually different ideal on $\omega \times \omega$ is defined by

$$\mathcal{ED} = \{A \subset \omega \times \omega : \exists m, n \in \omega \forall k > n\, (|\{l : \langle k, l\rangle \in A\}| \leq m)\}.$$

Define $\mathcal{ED}_{fin}$, an ideal on Δ by

$$\mathcal{ED}_{fin} = \{X \cap \Delta : X \in \mathcal{ED}\}$$

where $\Delta = \{\langle m, n\rangle : n \leq m\}$.

Theorem (Hrušák)

If $\mathcal{I}$ is a Borel ideal on ω, then $\text{non}^*(\mathcal{I}) = \omega$ or $\mathcal{ED}_{fin} \leq_K \mathcal{I}$.

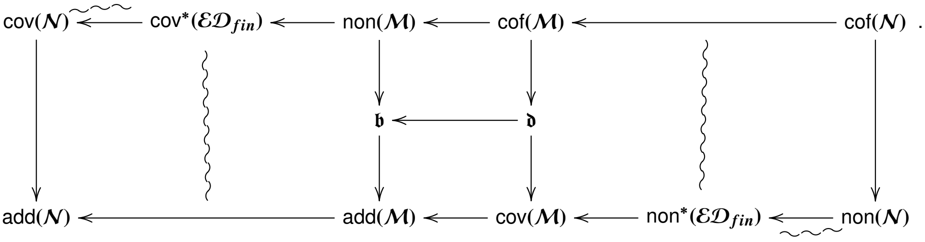
The uniformity number and the covering number of Borel ideals

Corollary

If $\mathcal{I}$ is a Borel ideal and $\mathrm{non}^*(\mathcal{I}) \neq \omega$, then $\mathrm{non}^*(\mathcal{ED}_{fin}) \leq \mathrm{non}^*(\mathcal{I})$ and $\mathrm{cov}^*(\mathcal{ED}_{fin}) \geq \mathrm{cov}^*(\mathcal{I})$

Proposition

1. $\min\{\mathfrak{d}, \mathrm{non}^*(\mathcal{ED}_{fin})\} = \mathrm{cov}(\mathcal{M})$.
2. $\max\{\mathfrak{b}, \mathrm{cov}^*(\mathcal{ED}_{fin})\} = \mathrm{non}(\mathcal{M})$.



Questions

Question

For each F_σ ideal $\mathcal{I}$ on ω,

1. $\mathit{Con}(\mathrm{cov}^*(\mathcal{I}) > \mathfrak{d})$?
2. $\mathit{Con}(\mathrm{non}^*(\mathcal{I}) < \mathfrak{b})$?
3. $\mathrm{non}^*(\mathcal{I}) \leq \mathfrak{l}$?
4. $\mathrm{cov}^*(\mathcal{I}) \geq \text{trans-add}(\mathcal{N})$?

The covering numbers of ideals on ω

Theorem (Laflamme)

If $\mathcal{I}$ is an F_σ-ideal on ω, then it is consistent that $\mathrm{cov}^*(\mathcal{I}) > \mathfrak{d}$.

$\mathcal{I}$ is P-ideal if

$$\forall \langle X_n : n \in \omega \rangle \subset \mathcal{I} \exists X \in \mathcal{I} \forall n \in \omega \, (X_n \subset^* X) .$$

Theorem

If $\mathcal{I}$ is an analytic P-ideal, then it is $F_{\sigma\delta}$.

Theorem (Hrušák and Hernández-Hernández)

If $\mathcal{I}$ is an analytic P-ideal on ω, then it is consistent that $\mathrm{cov}^*(\mathcal{I}) > \mathfrak{d}$.

The uniformity number of F_σ-ideals on ω

Theorem

If $\mathcal{I}$ is an F_σ-ideal on ω, then it is consistent that non*$(\mathcal{I}) < \mathfrak{b}$.

We shall show if $V \models$ CH, then $V^{\mathbb{L}_{\omega_2}} \models$ non*$(\mathcal{I}) < \mathfrak{b}$.

The Laver forcing

The Laver forcing $\mathbb{L}$ is defined by $T \in \mathbb{L}$ if

1. $T \subset \omega^{<\omega}$ is a tree and
2. $\forall s \in T\,(stem(T) \subset s \rightarrow |succ_T(s)| = \aleph_0)$.

$\mathbb{L}$ is ordered by inclusion.

Proposition

Let G be a $\mathbb{L}$-generic over V and $f_G = \bigcup\{stem(T) : T \in G\}$. Then $f_G \in \omega^\omega$ and $g \leq^* f_G$ for all $g \in \omega^\omega \cap V$.

$\mathbb{L}_{\omega_2}$ denotes the ω_2-stage countable support iteration of $\mathbb{L}$. Then

$$V^{\mathbb{L}_{\omega_2}} \models \mathfrak{b} = \omega_2.$$

The Laver property

Definition

A forcing notion $\mathbb{P}$ has the Laver property (LP) if for every $H : \omega \to \omega \in V$

$$\Vdash \forall f \in \prod_{n\in\omega} H(n) \cap V[\dot{G}] \exists A : \omega \to [\omega]^{<\omega} \in V$$
$$\forall n \in \omega \left(f(n) \in A(n) \land |A(n)| \leq 2^n \right).$$

Theorem

The LP is preserved under countable support iteration of proper forcing notions.

Theorem

The Laver forcing $\mathbb{L}$ has the LP.

So $\mathbb{L}_{\omega_2}$ has the LP.

Lower semi-continuous submeasure on ω and F_σ-ideal

Definition

$\varphi : \mathcal{P}(\omega) \to [0, \infty]$ is a submeasure if

1. $\varphi(X \cup Y) \leq \varphi(X) + \varphi(Y)$,
2. $X \subset Y \to \varphi(X) \leq \varphi(Y)$ and
3. $\varphi(n) < \infty$ for $n \in \omega$.

We say a submesure $\varphi : \mathcal{P}(\omega) \to [0, \infty]$ is lower semicontinuous if

4. $\varphi(X) = \lim_{n\to\infty} \varphi(X \cap n)$.

Theorem (Mazur)

If $\mathcal{I}$ is an F_σ-ideal on ω, then there exists a lower semicountinuous submeasure φ such that $\mathcal{I} = \{X \subset \omega : \varphi(X) < \infty\}$.

Proof of Main lemma 1

Lemma

Let $\mathcal{I}$ be an F_σ-ideal on ω. If $\mathbb{P}$ has the LP, then
$\Vdash_{\mathbb{P}}$ "$\forall X \in \mathcal{I} \cap V[\dot{G}] \exists A \in [\omega]^\omega \cap V\, (|X \cap A| < \aleph_0)$".

Proof.

Let $p \in \mathbb{P}$ and let $\dot{X}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\dot{X} \in \mathcal{I}$".
Let φ be a lower semicontinuous submeasure such that

$$\mathcal{I} = \{X \subset \omega : \varphi(X) < \infty\}.$$

Without loss of generality there exists $n \in \omega$ such that $p \Vdash_{\mathbb{P}}$ "$\varphi(\dot{X}) < n$".

Claim

For each $k \in \omega$ and $l \in \omega$, there exists $m \in \omega$ such that $\varphi([l, m]) > k$.

Let $\Pi = \langle I_j : j \in \omega \rangle$ be an interval partition of ω such that
$\varphi(I_j) > 2^j \cdot n$. □

Proof of main lemma 2

Proof.

Let $\Pi = \langle I_j : j \in \omega \rangle$ be an interval partition of ω such that $\varphi(I_j) > 2^j \cdot n$. By the LP for $\langle \dot{X} \restriction I_j : j \in \omega \rangle$, there are $q \le p$ and $A : \omega \to [\bigcup_{j \in \omega} 2^{I_j}]^{<\omega}$ in V such that for $j \in \omega$,

1. $A(j) \subset 2^{I_j}$, $|A(j)| \le 2^j$ and
2. $q \Vdash_{\mathbb{P}}$ "$\forall j \in \omega \left(\dot{X} \restriction I_j \in A(j) \right)$".

Without loss of generality $\varphi(J) \le n$ for $J \in A(j)$ and for $j \in \omega$. By the finite subadditivity of φ, for each $j \in \omega$,

$$\varphi(\bigcup A(j)) \le \sum_{J \in A(j)} \varphi(J) \le 2^j \cdot n.$$

So $I_j \setminus \bigcup A(j) \neq \emptyset$ for $j \in \omega$. Put $Y = \bigcup_{j \in \omega}(I_j \setminus \bigcup A(j)) \in [\omega]^\omega$. Then

$$q \Vdash_{\mathbb{P}} \text{“}\dot{X} \cap Y = \emptyset\text{”}.$$

Therefore $\Vdash_{\mathbb{P}}$ "$\forall X \in \mathcal{I} \exists Y \in [\omega]^\omega \cap V \, (|X \cap Y| < \aleph_0)$". □

Main Theorem

Lemma

Let $\mathcal{I}$ be an F_σ-ideal on ω. If $\mathbb{P}$ has the LP, then
$\Vdash_{\mathbb{P}}$ "$\forall X \in \mathcal{I} \cap V[\dot{G}] \exists A \in [\omega]^\omega \cap V\, (|X \cap A| < \aleph_0)$".

Theorem

If $\mathcal{I}$ is an F_σ-ideal on ω, then it is consistent that non$^*(\mathcal{I}) < \mathfrak{b}$.

Proof of theorem from lemma.

We know $V^{\mathbb{L}_{\omega_2}} \models \mathfrak{b} = \omega_2$.
Suppose $V \models CH$. By lemma, $[\omega]^\omega \cap V$ witnesses non$^*(\mathcal{I})$. So $V^{\mathbb{L}_{\omega_2}} \models$ non$^*(\mathcal{I}) = \omega_1$. Therefore $V^{\mathbb{L}_{\omega_2}} \models$ non$^*(\mathcal{I}) < \mathfrak{b}$.

□

$\mathfrak{l}$ and trans-add($\mathcal{N}$)

Definition (Kada)

Let $\mathcal{S} = \{\phi : \phi : \omega \to [\omega]^{<\omega} \wedge |\phi(n)| \leq n+1\}$.
For $H \in \omega^\omega$,

$$\mathfrak{l}_H = \min\{|\Phi| : \Phi \subset \mathcal{S} \wedge \forall f \in \Pi H(n) \exists \phi \in \Phi \forall^\infty n \in \omega (f(n) \in \phi(n))\}.$$

$$\mathfrak{l} = \sup\{\mathfrak{l}_h : h \in \omega^\omega\}.$$

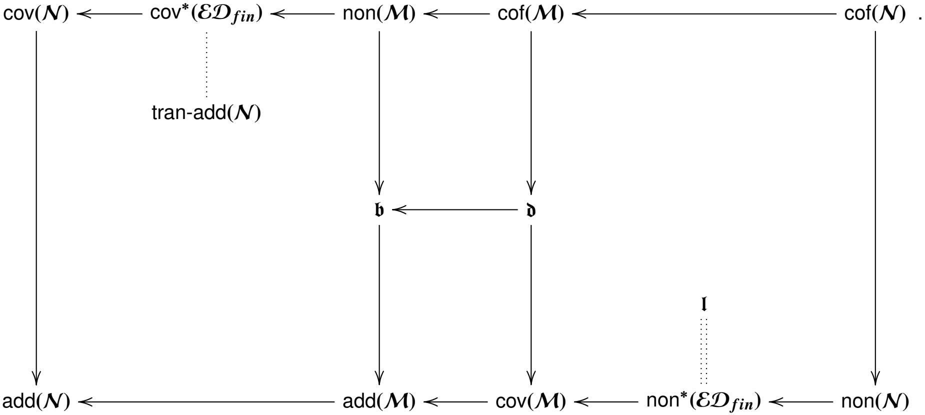
Theorem (Pawlikowski)

$$\text{trans-add}(\mathcal{N}) = \min\{|F| : F \subset \omega^\omega \wedge \exists H \in \omega^\omega (F \subset \Pi_{n\in\omega} H(n)) \wedge \forall \phi \in \mathcal{S} \exists f \in F \exists^\infty n \in \omega (f(n) \notin \phi(n))\}.$$

Corollary

If $\mathcal{I}$ is an F_σ-ideal on ω, then $\text{non}^*(\mathcal{I}) \leq \mathfrak{l}$ and $\text{cov}^*(\mathcal{I}) \geq$ trans-add($\mathcal{N}$).

Cichon's diagram



If $\mathcal{I}$ is an F_σ-ideal on ω and $\mathrm{non}^*(\mathcal{I}) \neq \omega$, then $\mathrm{non}^*(\mathcal{I})$ exists around :::.
If $\mathcal{I}$ is an F_σ-ideal on ω and $\mathrm{non}^*(\mathcal{I}) \neq \omega$, then $\mathrm{cov}^*(\mathcal{I})$ exists around ....

Reference

[1] Michael Hrušák and David Meza-Alcántara and Hiroaki Minami, “Around pair-splitting and pair-reaping”, preprint.

[2] Fernando Hernández-Hernández and Michael Hrušák, “Cardinal invariants of analytic P-ideals”, Canadian Journal of Mathematics,575–595, vol 59, Number 3, 2007.