Suslin forcing and parametrized ♦ principles

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Abstract

By using finite support iteration of Suslin forcing we construct several models which satisfy some ♦-like principles while other cardinal invariants are larger than ω₁.

1 Introduction

This work is about parametrized diamond principles, a broad framework of ♦-like principles introduced by Moore, Hrušák and Džamonja in [9] to analyze systematically ♦ and its consequences.

For our purpose call a triple (A, B, E) Borel invariant if |A|, |B| ≤ c, E ⊂ A × B, for each a ∈ A there exists b ∈ B such that (a, b) ∈ E, for each b ∈ B there exists a ∈ A such that (a, b) /∈ E and, A, B and E are Borel sets in some Polish space. If a triple (A, B, E) is a Borel invariant, then its evaluation ⟨A, B, E⟩ is given by

⟨A, B, E⟩ = min{|X| : X ⊂ B and ∀a ∈ A∃b ∈ X (aEb)}.

We call F : 2<ω₁ → A a Borel function if F↾ 2α is a Borel function for α < ω₁. Then ♦(A, B, E) is the following statement:

♦(A, B, E) For all Borel F : 2<ω₁ → A there exists g : ω₁ → B such that for every f : ω₁ → 2 the set {α ∈ ω₁ : F↾(f↾ α)Eg(α)} is stationary.

The witness g for given F in this statement will be called ♦(A, B, E)-sequence for F.

Note. When we deal with a Borel invariant whose evaluation is a well-known cardinal invariant, we will use the cardinal invariant to denote the Borel invariant (e.g., we will use ♦(add(ℕ)) to denote ♦(ℕ, ℕ, ⊈)).

In [9] Moore, Hrušák and Džamonja introduced several methods to construct parametrized diamond principles.
Theorem 1.1. [9] Let \( C(\omega_1) \) and \( B(\omega_1) \) be the Cohen and random forcing corresponding to the product space \( 2^{\omega_1} \). Then \( V^C(\omega_1) \models \Diamond(\text{non}(\mathcal{M})) \) and \( V^B(\omega_1) \models \Diamond(\text{non}(\mathcal{N})) \).

In [6] by using \( \omega_1 \)-stage finite support iteration several models which satisfy CH and some \( \Diamond(A, B, E) \) while others fail are constructed. For countable support iteration, there is a general theorem to construct \( \Diamond(A, B, E) \).

Theorem 1.2. [9] Suppose that \( \langle Q_\alpha : \alpha < \omega_2 \rangle \) is a sequence of Borel partial orders such that for each \( \alpha < \omega_2 \) \( Q_\alpha \) is equivalent to \( \wp(2) \times Q_\alpha \) as a forcing notion and let \( P_{\omega_2} \) be the countable support iteration of this sequence. If \( P_{\omega_2} \) is proper and \( (A, B, E) \) is a Borel invariant then \( P_{\omega_2} \) forces \( \langle A, B, E \rangle \leq \omega_1 \) iff \( P_{\omega_2} \) forces \( \Diamond(A, B, E) \).

This result is best possible because the following proposition holds.

Proposition 1.3. Let \( (A, B, E) \) be a Borel invariant. If \( \Diamond(A, B, E) \) holds, then \( \langle A, B, E \rangle \leq \omega_1 \).

In this paper we shall prove the consistency of \( \Diamond(\tau) + \eta = \omega_2 \) for several pairs \( (\tau, \eta) \) of cardinal invariants of the continuum. As mentioned above (Theorem 1.2) this has been achieved before by Moore, Hrušák and Džamonja in [9]. They used countable support iteration to show \( \Diamond(\tau) + \eta = \omega_2 \). But our approach is completely different from the methods of Moore, Hrušák and Džamonja. We shall use finite support iteration of Suslin c.c.c forcing notions to prove the consistency of \( \Diamond(\tau) + \eta = \omega_2 \).

And our results are more general. We can obtain the consistency of \( \Diamond(\tau) + \eta = \kappa \) not just \( \Diamond(\tau) + \eta = \omega_2 \).

Along the way, new preservation results for finite support iteration are established. These are interesting in their own right.

The present paper is organized as follows. Section 2 shows some properties of Suslin forcing. Section 3 presents several models satisfying parametrized diamond principles by using \( \omega_2 \)-stage finite support iteration of Suslin forcing notions.

2 Suslin c.c.c forcing and complete embedding

In this section we will study some properties of a family of c.c.c forcing notions which have a nice definition.

Definition 2.1. [1, p.168] A forcing notion \( P = (P, \leq_P) \) has a Suslin definition if \( P \subset \omega_\omega \), \( \leq_P \subset \omega_\omega \times \omega_\omega \) and \( \bot_P \subset \omega_\omega \times \omega_\omega \) are \( \Sigma^1_1 \).

\( P \) is Suslin if \( P \) is c.c.c and has a Suslin definition.

Definition 2.2. [1, p.168] Let \( M \models \text{ZFC}^* \). A Suslin forcing \( P \) is in \( M \) if all the parameters used in the definitions of \( P \), \( \leq_P \) and \( \bot_P \) are in \( M \).

We will interpret Suslin forcing notion in forcing extensions using its code rather than taking the ground model forcing notion.
Definition 2.3. Let $\mathbb{A}$ and $\mathbb{B}$ be forcing notions. Then $i : \mathbb{A} \rightarrow \mathbb{B}$ is a complete embedding if

1. for all $a, a' \in \mathbb{A}$ if $a \leq a'$, then $i(a) \leq i(a')$,
2. for all $a_1, a_2 \in \mathbb{A}$ if $a_1 \perp a_2$ then $i(a_1) \perp i(a_2)$ and
3. for all $A \subset P$ if $A$ is a maximal antichain in $\mathbb{A}$ then $i[A]$ is a maximal antichain in $\mathbb{B}$.

If there is a complete embedding from $\mathbb{A}$ to $\mathbb{B}$ then we write $\mathbb{A} \preceq \mathbb{B}$.

Lemma 2.4. Assume $\mathbb{A} \preceq \mathbb{B}$ and $P$ is a Souslin forcing notion. Then $\mathbb{A} \ast \dot{P} \preceq \mathbb{B} \ast P$ where $P$ are names for interpretation of the code for the Souslin forcing notion in each model.

Proof of Lemma. Let $i : \mathbb{A} \rightarrow \mathbb{B}$ be a complete embedding. Define $\dot{i} : \mathbb{A} \ast \dot{P} \rightarrow \mathbb{B} \ast P$ by $i(\dot{a}, \dot{x}) = (i(a), i_*(\dot{x}))$ where $i_*$ is the class map from $\mathbb{A}$-names to $\mathbb{B}$-names induced by $i$ (see [7, p.222]). We will show if $A \subset \mathbb{A} \ast \dot{P}$ is a maximal antichain, then $i[A]$ is also a maximal antichain. It is clear $i[A]$ is an antichain. Let $A$ be a maximal antichain of $\mathbb{A} \ast \dot{P}$ and put $A = \{(a_\alpha, \dot{p}_\alpha) : \alpha < \kappa \}$. Assume there exists $(b, \dot{p}) \in \mathbb{B} \ast P$ such that $(b, \dot{p})$ and $i((a_\alpha, \dot{p}_\alpha))$ are incompatible for all $\alpha < \kappa$. Let $G$ be a $(\mathbb{B}, V)$-generic such that $b \in G$ and let $H = i^{-1}[G]$. Let $A' = \{\dot{p}_\alpha[H] : i(a_\alpha) \in G\} \in V[H]$.

Subclaim 2.4.1. $V[H] \models \langle A' \text{ is a maximal antichain} \rangle$.

Proof of subclaim. Firstly we shall show $A'$ is an antichain. Suppose $\dot{p}_\alpha[H], \dot{p}_\beta[H] \in A'$. Since $a_\alpha, a_\beta \in H$, $a_\alpha$ and $a_\beta$ are compatible. Since $(a_\alpha, \dot{p}_\alpha)$ and $(a_\beta, \dot{p}_\beta)$ are incompatible, for all $r \leq a_\alpha, a_\beta$ there exists $s \leq r$ such that $s \not\models \langle \dot{p}_\alpha \text{ and } \dot{p}_\beta \text{ are incompatible} \rangle$. So $\dot{p}_\alpha[H]$ and $\dot{p}_\beta[H]$ are incompatible. Hence $A'$ is an antichain.

From now on we shall show maximality of $A'$. Assume to the contrary, there exists $p \in P$ such that $p$ and $\dot{p}_\alpha[H]$ are incompatible for any $\dot{p}_\alpha[H] \in A'$. So there exists $a \in H$ and an $\mathbb{A}$-name $\dot{P}$ for $p$ such that $a \models \forall \alpha < \kappa (a_\alpha \in H \rightarrow \dot{p}$ and $\dot{p}_\alpha$ are compatible ). Hence $\langle a, \dot{p} \rangle$ and $\langle a_\alpha, \dot{p}_\alpha \rangle$ are incompatible. But it contradicts the maximality of $A$.

[Subclaim]

$V[H] \models \langle A' \text{ is a maximal antichain in } \mathbb{P} \rangle$ and $\langle A' \text{ is a maximal antichain in } \mathbb{P}' \rangle$ is a $\Pi^1_1$ statement with parameter $A'$, $\mathbb{P} \leq \mathbb{P}'$ and $\bot$. Hence by $\Pi^1_1$ absoluteness $V[G] \models \langle A' \text{ is a maximal antichain in } \mathbb{P} \rangle$ But this is a contradiction to the fact $V[G] \models \langle \dot{p}[G] \perp i_*(\dot{p}_\alpha)[G] \rangle$ for $i(a_\alpha) \in G$.

□

Theorem 2.5. Let $\langle Q_\alpha : \alpha < \kappa \rangle$ be a sequence of Souslin forcing notions. Let $\mathbb{P}_\kappa$ be the limit of the finite support iteration of $\langle \mathbb{P}_\alpha, Q_\alpha : \alpha < \kappa \rangle$. Then $\mathbb{A} \preceq \mathbb{B}$ implies $\mathbb{A} \ast \dot{P}_\kappa \preceq \mathbb{B} \ast \dot{P}_\kappa$.

Proof. By induction on $\kappa$. Limit stage is clear. Successor stage follows from above Lemma.
Corollary 2.6. Let \( (Q_\alpha : \alpha < \kappa) \) be a sequence of Suslin forcing notions. Let \( I \subset \kappa \). Then \( P_I \prec P_\kappa \) where \( P_I \) is the limit of the iteration of \( \langle P_\alpha , R_\alpha : \alpha < \kappa \rangle \) where \( \Vdash_{P_I} \hat{R}_\alpha = \begin{cases} \hat{Q}_\alpha & \alpha \in I \\ \{1\} & \text{otherwise}. \end{cases} \)

3 Construction of Parametrized \( \Diamond \) principles

We shall construct several models by finite support iteration of Suslin forcing notions.

If two Borel invariants \((A_1, B_1, E_1), (A_2, B_2, E_2)\) are comparable in the Borel Tukey order, then \( \Diamond(A_1, B_1, E_1) \) and \( \Diamond(A_2, B_2, E_2) \) have some relation:

Definition 3.1. (Borel Tukey ordering [3]) Given a pair of Borel invariants \((A_1, B_1, E_1), (A_2, B_2, E_2)\), we say that \( (A_1, B_1, E_1) \leq_{BT} (A_2, B_2, E_2) \) if there exist Borel maps \( \phi : A_1 \to A_2 \) and \( \psi : B_2 \to B_1 \) such that \((\phi(a), b) \in E_2 \) implies \((a, \psi(b)) \in E_1 \).

Proposition 3.2. [9] Let \((A_1, B_1, E_1), (A_2, B_2, E_2)\) be Borel invariants. Suppose \((A_1, B_1, E_1) \leq_{BT} (A_2, B_2, E_2)\) and \( \Diamond(A_2, B_2, E_2) \) holds. Then \( \Diamond(A_1, B_1, E_1) \) holds.

Concerning \( \leq_{BT} \), we know the following diagram holds.

(Cichoń’s diagram)

\[ \begin{array}{cccc}
(R, N, \in) & \leftarrow & (M, R, \notin) & \leftarrow (M, C) \leftarrow (N, C) \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow
\end{array} \]

\[ \begin{array}{cccc}
\langle \omega^\omega, \not<\star \rangle & \leftarrow & \langle \omega^\omega, \leq \star \rangle
\end{array} \]

\[ \begin{array}{cccc}
\langle N, \notin \rangle & \leftarrow & \langle M, \notin \rangle & \leftarrow \langle R, M, \in \rangle \leftarrow \langle N, R, \notin \rangle
\end{array} \]

(The direction of the arrow is from larger to smaller in the Borel Tukey order). Hence the following holds:

\[ \begin{array}{cccc}
\Diamond(\text{cov}(N)) & \leftarrow & \Diamond(\text{non}(M)) & \leftarrow \Diamond(\text{cof}(M)) \leftarrow \Diamond(\text{cof}(N)) \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow
\end{array} \]

\[ \begin{array}{cccc}
\Diamond(\text{b}) & \leftarrow & \Diamond(\text{b})
\end{array} \]

\[ \begin{array}{cccc}
\Diamond(\text{add}(N)) & \leftarrow & \Diamond(\text{add}(M)) & \leftarrow \Diamond(\text{cof}(M)) \leftarrow \Diamond(\text{non}(N))
\end{array} \]
We call this diagram “Cichoń’s diagram for parametrized diamonds”. We will deal with Borel invariants in Cichoń’s diagram.

**Definition 3.3.**  
1. The Hechler forcing notion is defined as follows:
   \[
   \langle s, f \rangle \in D \text{ if } s \in \omega^{<\omega}, f \in \omega^\omega \text{ and } s \subset f.
   \]
   It is ordered by
   \[
   \langle s, f \rangle \leq \langle t, g \rangle \text{ if } s \supset t \text{ and } g \leq f.
   \]
2. The eventually different forcing notion is defined as follows:
   \[
   \langle s, H \rangle \in E \text{ if } s \in \omega^{<\omega} \text{ and } H \in [\omega^\omega]^{<\omega}.
   \]
   It is ordered by \( \langle s, H \rangle \leq \langle t, G \rangle \) if \( s \supset t \), \( H \supset G \) and
   \[
   \text{for all } g \in G \text{ for all } j \in [|t|, |s|) \begin{cases} \text{s}(j) \neq g(j). \end{cases}
   \]
3. Let \( \text{Borel}(2^\omega) \) be the smallest \( \sigma \)-algebra containing all open subsets of \( 2^\omega \). Let \( \mu \) be the standard product measure on \( 2^\omega \) and let \( N = \{ A \in \text{Borel}(2^\omega) : \mu(A) = 0 \} \). For \( A, B \in \text{Borel}(2^\omega) \) let \( A \equiv_N B \) if \( A \triangle B \in N \). Let \( \mathcal{N} \) be the equivalence class of the set \( A \) with respect to the equivalence relation \( \equiv_N \).
   Define
   \[
   \mathcal{B} = \{ [A]_\mathcal{N} : A \in \text{Borel}(2^\omega) \}.
   \]
   It is ordered by \( [A]_\mathcal{N} \leq [B]_\mathcal{N} \text{ if } A \setminus B \in \mathcal{N} \).

**Theorem 3.4.** Let \( \kappa \) be an ordinal with \( \text{cf}(\kappa) > \omega_1 \). Let \( D_\kappa, E_\kappa, B_\kappa \) and \( (B \ast D)_\kappa \) be the \( \kappa \)-stage finite support iteration of \( D, E, B \) and \( B \ast D \) respectively. Then the following statements hold:

1. \( V^{D_\kappa} \models \diamondsuit(\text{cov}(\mathcal{N})) \).
2. \( V^{E_\kappa} \models \diamondsuit(\text{cov}(\mathcal{N})) \text{ and } \diamondsuit(b) \).
3. \( V^{B_\kappa} \models \diamondsuit(b) \).
4. \( V^{(B \ast D)_\kappa} \models \diamondsuit(\text{add}(\mathcal{N})) \).

**Proof.** (1) Let \( \Pi \) be a partition of \( \omega \) into finite intervals \( I_n \) with \( |I_n| = n + 1 \) for \( n \in \omega \). Define a relation \( =^\omega \) so that \( x =^\omega y \) if there exist infinitely many \( n \in \omega \) such that \( x \upharpoonright I_n = y \upharpoonright I_n \). We will show \( V^{D_\kappa} \models \diamondsuit(2^\omega \upharpoonright I_n) \). Let \( \bar{F} \) be a \( D_\kappa \)-name such that \( \Vdash_{D_\kappa} \bar{F} : 2^{<\omega_1} \to 2^\omega \). Since \( D_\kappa \) has the c.c.c., a real \( \bar{r}_\alpha \) coding the Borel function \( \bar{F} \upharpoonright 2^\alpha \) appears at an intermediate stage. By \( \text{cf}(\kappa) > \omega_1 \) we can assume \( \bar{F} \) is a \( D_\beta \)-name for some \( \beta < \kappa \). Furthermore we can assume \( \bar{F} \) is a Borel function in ground model. Let \( \bar{F} \) be a Borel function in ground model. Let \( \bar{f} \) be a \( D_\kappa \)-name such that \( \Vdash_{D_\kappa} \bar{f} : \omega_1 \to 2 \). Then the following claim holds:
Claim 3.4.1. Define \( C_f \subset \omega_1 \) by

\[ C_f = \{ \alpha < \omega_1 : \dot{f} \restriction \alpha \text{ is } D_{\omega_1 \cup \omega_1, \kappa}\text{-name} \} \]

Then \( C_f \) contains a club.

Remark 3.4.2. More precisely we should write

\[ C_f = \{ \alpha < \omega_1 : \text{there exists } D_{\omega_1 \cup \omega_1, \kappa}\text{-name } \dot{x}_\alpha \text{ such that } \Vdash_{D_n} \dot{f} \restriction \alpha = i_\ast(\dot{x}_\alpha) \} \]

where \( i_\ast \) is a class function from \( D_{\omega_1 \cup \omega_1, \kappa}\)-names to \( \kappa\)-names induced by the complete embedding \( i : D_{\omega_1 \cup \omega_1, \kappa} \prec D_\kappa \). But for convenience we will think of a \( D_\kappa\)-name \( \dot{x} \) as \( D_1\)-name if there exists a \( D_1\)-name \( y \) such that \( \Vdash_{D_n} \dot{x} = i_\ast(y) \) where \( i_\ast \) is a complete embedding from \( D_1 \) to \( D_\kappa \).

For \( \alpha \in C_f \) let \( \dot{x}_\alpha \) be a \( D_{\omega_1 \cup \omega_1, \kappa}\)-name such that \( \Vdash_{\omega_1 \cup \omega_1} F(\dot{f} \restriction \alpha) = \dot{x}_\alpha \).
Let \( \dot{c}_\alpha \) be a \( D_{\omega_1} \)-name such that for all \( \dot{x} \in 2^{\omega_1} \cap \V^\omega_1 \) \( \Vdash_{D_{\omega_1 \cup \omega_1}} 3^{\infty}_n (\dot{c}_\alpha \restriction I_n = \dot{x} \restriction I_n) \).
We can obtain such \( \dot{c}_\alpha \). For example put \( \dot{c}_\alpha \) a \( D_{\omega_1} \)-name for a Cohen real over \( \V^\omega_1 \).

We shall show \( \V^\omega_1 \) \( 3^{\infty}_n (\dot{c}_\alpha \restriction I_n = \dot{x}_\alpha \restriction I_n) \). To prove this we will work in \( \V^\omega_1 \) and show the following lemma.

Lemma 3.5. Suppose \( \gamma \) is an ordinal and \( P \) is a forcing notion which has a \( P \)-name \( \dot{c} \) such that for all \( x \in 2^{\omega_1} \cap V \) \( \Vdash_P 3^{\infty}_n (x \restriction I_n = \dot{c} \restriction I_n) \).
Let \( \dot{x} \) be a \( \omega_1 \)-name such that \( \Vdash_{\omega_1 \cup \omega_1} 3^{\infty}_n \dot{x} \restriction I_n = \dot{x} \restriction I_n \).
Here precisely we should write \( \Vdash_{\omega_1 \cup \omega_1} 3^{\infty}_n (\dot{c} \restriction I_n = i_\ast(\dot{x}) \restriction I_n) \) where \( i_\ast \) is a canonical map from \( \omega_1 \)-names to \( P \)-names induced by the complete embedding \( i : D_\gamma \to P \).

Proof. We proceed by induction on \( \gamma \).

First step
Let \( \dot{x} \) be a \( D_{\gamma} \)-name such that \( \Vdash_D \dot{x} \in 2^{\omega_1} \). Let \( \dot{c} \) be a \( P \)-name such that \( \Vdash_P " 3^{\infty}_n \in \omega (\dot{c} \restriction I_n = x \restriction I_n) " \) for all \( x \in V \cap 2^{\omega_1} \). Let \( (p_0, q_0) \in P \times D_\gamma \) and \( \gamma \in \omega \).

It suffices to show there exist \( (p_1, q_1) \leq_{\omega_1 \cup \omega_1} (p_0, q_0) \) and \( \gamma \geq \gamma \) such that \( \langle p_1, q_1 \rangle \Vdash_{P_{\omega_1 \cup \omega_1}} \dot{x} \restriction I_n = \dot{c} \restriction I_n \).

Without loss of generality we can assume \( p_0 \Vdash q_0 = (s, \dot{g}) \) for some \( s \in \omega^{<\omega} \).
Let \( x_s \in V \cap 2^{\omega_1} \) such that

\[ \forall j \in \omega \forall g' \in \omega^{<\omega} (g' \supset s \rightarrow \neg\langle s, g' \rangle \Vdash_D \dot{x} \restriction I_j \neq x_s \restriction I_j) \].

Let \( r \leq p_0 \) such that \( r \Vdash_{\omega_1 \cup \omega_1} x_s \restriction I_n = \dot{c} \restriction I_n \) for some \( \gamma \geq \gamma \). Then define \( \langle r_k : k \in \omega \rangle \) a decreasing sequence of \( P \) and \( g^* \in 2^{\omega_1} \cap V \) such that \( r_0 \leq_P r \) and \( r_k \Vdash_P \dot{g} \restriction (|s| + k) = g^* \restriction (|s| + k) \).

By definition of \( x_s \) there is \( (t, h) \leq_D (s, g^*) \) such that \( (t, h) \Vdash_D x_s \restriction I_n = \dot{x} \restriction I_n \).

Since \( (t, h) \leq_D (s, g^*) \), for all \( i \in |s|, |t| \) \( t(i) \geq g^*(i) \). Since \( r_i \Vdash_P \forall i \in |t| \langle \hat{g}(i) = g^*(i) \leq t(i) \rangle \) and \( \forall \dot{g} \restriction (s, \dot{g}) \) are compatible. Put \( p_1 = r_i \) and put a \( P \)-name \( \dot{q}_i \) so that \( p_1 \Vdash_P \dot{q}_i \leq_D (s, \hat{g}, (t, h)) \). Then \( (p_1, q_1) \leq_{\omega_1 \cup \omega_1} (p_0, q_0) \) and \( p_1 \Vdash_{\omega_1 \cup \omega_1} x_s \restriction I_n = \dot{c} \restriction I_n \) by \( p_1 \Vdash_{\omega_1 \cup \omega_1} \dot{x} \restriction I_n = x_s \restriction I_n \).

Therefore \( (p_1, q_1) \Vdash_{P_{\omega_1 \cup \omega_1}} \dot{x} \restriction I_n = x_s \restriction I_n = \dot{c} \restriction I_n \).
Successor step:
Suppose the lemma holds for $\gamma$. Let $\dot{x}$ be a $\mathbb{D}_{\gamma+1}$-name such that $\models_{\mathbb{D}_{\gamma+1}} \dot{x} \in 2^\omega$.
Let $(p_0, \dot{q}_0) \in \mathbb{P} \ast \dot{\mathbb{D}}_{\gamma+1}$ and $m \in \omega$. Without loss of generality we can assume $(p_0, \dot{q}_0 \mid \gamma) \models_{\mathbb{P} \ast \mathbb{D}_{\gamma}} \dot{q}_0(\gamma) = \langle \dot{s}, \dot{y} \rangle$ for some $s \in \omega^\omega$.
Let $\dot{x}_s$ be a $\mathbb{D}_\gamma$-name such that
$$\models_{\mathbb{D}_\gamma} \forall j \in \omega \forall \dot{y}' \in \omega^\omega \langle \dot{y}' \supset \dot{s} \rightarrow \neg \langle \dot{s}, \dot{y}' \rangle \models_{\mathbb{D}_\gamma} \dot{x}_s \mid I_j \neq \dot{x} \mid I_j \rangle.$$ 
By induction hypothesis there exists $\langle p, q, \dot{\gamma} \rangle \models_{\mathbb{P} \ast \mathbb{D}_\gamma} \dot{q}(\gamma) = \langle \dot{s}, \dot{y} \rangle$ for some $s \in \omega^\omega$.
Since $\mathbb{D}_\gamma < \mathbb{P} \ast \dot{\mathbb{D}}_\gamma$, there is a $\mathbb{D}_\gamma$-name $\dot{Q}$ for a partial order such that $\models_{\mathbb{P} \ast \dot{\mathbb{D}}_\gamma} \mathbb{D}_\gamma \ast \dot{Q}$. Let $q^*$ be the projection of $\langle p', q' \rangle$ to $\mathbb{D}_\gamma$.
Define $\mathbb{D}_\gamma$-names $\dot{y}^*$ and $\langle \dot{r}_k : k \in \omega \rangle$ such that
(i) $\models_{\mathbb{D}_\gamma} \dot{y}^* \in \omega^\omega$ and $\dot{r}_k \in \dot{Q}$ for $k \in \omega$,
(ii) $(q^*, \dot{r}_0) \leq (p', q')$,
(iii) $\models_{\mathbb{D}_\gamma} \dot{r}_{k+1} \leq \dot{Q} \dot{r}_k$ for $k \in \omega$ and
(iv) $\models_{\mathbb{D}_\gamma} \langle q^* \models_{\dot{Q}} \dot{y}(k) = \dot{y}^*(k) \rangle$.
Let $q^*_1 \models_{\mathbb{D}_\gamma} q^*$ such that there exists $t \in \omega^\omega$ and $\mathbb{D}_\gamma$-name $\dot{h}$ for a function from $\omega$ to $\omega$ such that $q^*_1 \models_{\mathbb{D}_\gamma} \langle \langle \dot{s}, \dot{y}^* \rangle, (\dot{t}, \dot{h}) \rangle \models_{\mathbb{D}_\gamma} \dot{x}_s \models I_n = \dot{x} \models I_n \rangle$.
Since $\langle q^*_1, \dot{r}_1 \rangle \models_{\mathbb{D}_\gamma} \langle \langle \dot{s}, \dot{y}^*, \dot{r} \rangle, (\dot{t}, \dot{h}) \rangle \models_{\mathbb{D}_\gamma} \dot{x} \models I_n = \dot{x} \models I_n \rangle$.
Therefore $(p_1, q_1) \models_{\mathbb{P} \ast \mathbb{D}_{\gamma+1}} \dot{c} \mid I_n = \dot{x} \models I_n$.
Limit step:
Suppose $\gamma$ is a limit ordinal and for $\beta < \gamma$ the lemma holds. Without loss of generality we can assume the cofinality of $\gamma$ is $\omega$. Let $\langle \gamma_i : i \in \omega \rangle$ be a strictly increasing sequence converging to $\gamma$. Let $(p_0, \dot{q}_0) \in \mathbb{P} \ast \dot{\mathbb{D}}_\gamma$, $m \in \omega$ and $\dot{x}$ be a $\mathbb{D}_\gamma$-name such that $\models_{\mathbb{D}_\gamma} \dot{x} \in 2^\omega$. Suppose $(p_0, \dot{q}_0) \in \mathbb{P} \ast \mathbb{D}_\gamma$.
In $V^{\mathbb{D}_\gamma}$ let $\langle r_k : k \in \omega \rangle$ be a decreasing sequence of $\mathbb{D}_{\gamma+\gamma}$ such that $r_k \mid_{\mathbb{D}_{\gamma+\gamma}} \langle \dot{x} \mid I_k = x_j \mid I_k \rangle$ where $x_j \in 2^\omega \cap V^{\mathbb{D}_\gamma}$.
Back into $V$ let $\dot{r}_k$ and $\dot{x}_j$ be $\mathbb{D}_{\gamma+\gamma}$-names such that $\models_{\mathbb{D}_{\gamma+\gamma}} \langle \dot{r}_k : k \in \omega \rangle$ and $\dot{x}_j$ satisfies the above.
By induction hypothesis there exists $\langle p', q' \rangle \models_{\mathbb{P} \ast \mathbb{D}_\gamma} \langle p_0, \dot{q}_0 \rangle$ and $m \geq n$ such that $(p', q') \models_{\mathbb{P} \ast \mathbb{D}_\gamma} \dot{c} \models I_n = \dot{x}_j \models I_n$. Put $p_1 = p'$ and $\models_{\mathbb{P}} \dot{q}_1 = \dot{q}' \models \dot{r}_n$.
Then $(p_1, q_1) \models_{\mathbb{P} \ast \mathbb{D}_\gamma} \langle q_1 \models_{\mathbb{D}_\gamma} \dot{c} \models I_n = \dot{x}_j \models I_n = \dot{x} \models I_n \rangle$.

Lemma □
Let $\dot{c}_\alpha$ be a $\mathbb{D}_{\omega_1}$-name such that $\models_{\mathbb{D}_{\omega_1}} \exists^\infty n \left( \dot{c}_\alpha \upharpoonright I_n = \dot{x} \upharpoonright I_n \text{ for } \dot{x} \in 2^\omega \cap V^{\mathbb{D}_{\omega_1}} \right)$. By the above lemma if $\alpha \in C_f$, then $\models_{\mathbb{D}_{\omega_1}} \exists^\infty n \left( \dot{x}_\alpha \upharpoonright I_n = F(\dot{f} \upharpoonright \alpha) \upharpoonright I_n = \dot{c}_\alpha \upharpoonright I_n \right)$. Hence $\models_{\mathbb{D}_{\omega_1}} \langle \dot{c}_\alpha : \alpha \in \omega_1 \rangle$ is a $\langle 2^\omega, =_{\text{H}}^\mathbb{D} \rangle$-sequence for $F$.

Let $\phi : 2^\omega \to \mathcal{N}$ be the function such that

$$\phi(x) = \{ y \in 2^\omega : \exists^\infty n \left( x \upharpoonright I_n = y \upharpoonright I_n \right) \}.$$ 

Then $\phi : 2^\omega \to \mathcal{N}$ and the identity function $\text{id} : 2^\omega \to 2^\omega$ witness $(2^\omega, \mathcal{N}, \epsilon) \leq_B (2^\omega, =_{\text{H}}^\mathbb{D})$ (see [2, Theorem 5.11]). So $V^{\mathbb{D}_{\omega_1}} \models \langle 2^\omega, \mathcal{N}, \epsilon \rangle$.

(1) $\square$

(2) $\models_{\mathbb{E}_\gamma} \check{\diamond} (\text{cov}(\mathcal{N}))$ is similar to (1). We shall only show $\models_{\mathbb{E}_\gamma} \check{\diamond} (\omega^\omega, \check{x}^*)$. To prove this it suffices to show the following lemma:

**Lemma 3.6.** Suppose $\gamma$ is an ordinal and $\mathbb{P}$ is a forcing notion which has a $\mathbb{P}$-name $\check{c}$ such that for all $x \in \omega^\omega \cap V \models \exists^\infty n (x(n) < \check{c}(n))$. Let $\check{x}$ be a $\mathbb{E}_\gamma$-name such that $\models_{\mathbb{E}_\gamma} \check{x} \in \omega^\omega$. Then $\models_{\mathbb{P} \ast \check{c}} \exists^\infty n (\check{x}(n) < \check{c}(n))$.

**Proof.** We proceed by induction on $\gamma$. We shall only prove the successor step. The rest of the proof is similar to the proof of Lemma 3.5.

**Successor step:**

Suppose the lemma holds for $\gamma$. Let $\check{x}$ be a $\mathbb{E}_{\gamma+1}$-name such that $\models_{\mathbb{E}_{\gamma+1}} \check{x} \in \omega^\omega$. Let $(p_0, \check{q}_0) \in \mathbb{P} \ast \check{c}_{\gamma+1}$ and $m \in \omega$. Without loss of generality we can assume $(p_0, \check{q}_0 \upharpoonright \gamma) \models_{\mathbb{P} \ast \check{c}_{\gamma}} "\check{q}_0(\gamma) = (s, \check{F})"$ and $\check{F} = \{ \check{f}_j : j < l \}^\gamma$ for some $l \in \omega$ and $s \in \omega^\omega$. Let $\check{x}_{s,l}$ be a $\mathbb{E}_\gamma$-name such that

$$\models_{\mathbb{E}_\gamma} \check{x}_{s,l}(i) = \min \{ j : \forall \check{H} \subseteq \omega^\omega \text{ with } |\check{H}| = l \left( \gamma(s, \check{H}) \models \check{x}(i) = j \right) \}.$$ 

By induction hypothesis there is $(p', \check{q}') \in \mathbb{P} \ast \check{c}_{\gamma}$ and $n \geq m$ such that $(p', \check{q}') \leq_{\mathbb{P} \ast \check{c}_{\gamma}} (p_0, \check{q}_0 \upharpoonright \gamma)$ and $(p', \check{q}') \models_{\mathbb{P} \ast \check{c}_{\gamma}} \check{c}(n) > \check{x}_{s,l}(n)$. Since $\mathbb{E}_\gamma < \mathbb{E}_{\gamma+1}$, there is a $\mathbb{E}_\gamma$-name $\check{Q}$ for a partial order such that $\mathbb{P} \ast \check{c}_{\gamma} \equiv \mathbb{E}_\gamma \ast \check{Q}$. Let $q^*$ be a projection of $(p', \check{q}')$ to $\mathbb{E}_\gamma$. Find $\mathbb{E}_\gamma$-names $\langle \check{r}_k : k \in \omega \rangle$ and $\check{F}^*$ such that

(i) $\models_{\mathbb{E}_\gamma} \check{F}^* = \{ \check{f}_j : j < l \} \subseteq \omega^\omega$ and $\check{r}_k \in \check{Q}$ for $k \in \omega$,

(ii) $(q^*, \check{r}_0) \leq (p', \check{q}')$,

(iii) $\models_{\mathbb{E}_\gamma} \check{r}_{k+1} \leq_{\check{Q}} \check{r}_k$ for $k \in \omega$ and,

(iv) $(q^*, \check{r}_k) \models_{\mathbb{E} \ast \check{Q}} \forall j < l \left( \check{f}_j(k) = \check{f}_j^*(k) \right)$ for $k \in \omega$.

Then there are $q^*_t \leq_{\mathbb{E}_\gamma} q^*$, $t \in \omega^{<\omega}$ and $\mathbb{E}_\gamma$-name $\check{G}$ such that $q_t^* \models_{\mathbb{E}_\gamma} "(t, \check{G}) \leq_{\check{Q}} (s, \check{F})"$ and $(t, \check{G}) \models_{\mathbb{E}_\gamma} \check{x}(n) \leq \check{x}_{s,l}(n)$.

Since $(q^*, \check{r}_|t|) \models_{\mathbb{E} \ast \check{Q}} \forall j < l \forall k < |t| \left( \check{f}_j(k) = \check{f}_j^*(k) \right)$ and $q^*_t \models_{\mathbb{E}_\gamma} \forall j < n \forall k \in [|s|, |t|] \left( \check{f}_j^*(k) \neq t(k) \right)$, $(q^*_t, \check{r}_|t|) \models_{\mathbb{E} \ast \check{Q}} (t, \check{G})$ and $(s, \check{F})$ are compatible.
Suppose positive finitely additive measure \( \gamma \) is an ordinal and \( \mathbb{P} \) is a forcing notion which has a \( \mathbb{P} \)-name \( c \) such that for all \( x \in \omega^\omega \cap V \models \exists n (x(n) < c(n)) \). Let \( \check{x} \) be a \( \mathbb{B}_\gamma \)-name such that \( \models \check{x} \in \omega^\omega \). Without loss of generality we can assume \( \check{x} \in \omega^\omega \). Therefore \( \check{x} < c(n) \).

\[ \square \]

(3) To prove (3) it suffices to show the following lemma:

**Lemma 3.7.** Suppose \( \gamma \) is an ordinal and \( \mathbb{P} \) is a forcing notion which has a \( \mathbb{P} \)-name \( c \) such that for all \( x \in \omega^\omega \cap V \models \exists n (x(n) < c(n)) \). Let \( \check{x} \) be a \( \mathbb{B}_\gamma \)-name such that \( \models \check{x} \in \omega^\omega \). Without loss of generality we can assume \( \check{x} \in \omega^\omega \).

**Proof of lemma.** We proceed by induction on \( \gamma \). We shall prove only the successor step.

**Successor step:**
Suppose for \( \gamma \) the lemma holds. Let \( \mu \) be a measure on \( \mathbb{B}_\gamma \). Let \( \check{x} \) be a \( \mathbb{B}_{\gamma+1} \)-name such that \( \models \check{x} \in \omega^\omega \). Let \( \check{x}^+ \) be a \( \mathbb{B}_\gamma \)-name such that

\[ \models (\check{x}(k) \leq \check{x}^+(k)) \models \mathbb{B} \geq 1 - \frac{1}{2^k}. \]

Let \( (p_0, q_0) \in \mathbb{P} * \mathbb{B}_{\gamma+1} \) and \( m \in \omega \). Without loss of generality we can assume \( (p_0, q_0 \mid \gamma) \models \mathbb{P} \mathbb{B} \models \mu(q_0(\gamma)) \geq \frac{1}{2} \). By induction hypothesis there is \( (p', q') \in \mathbb{P} * \mathbb{B}_\gamma \) such that \( (p', q') \models (p, q \mid \gamma) \) and \( n \geq m, l \) such that

\[ \check{x}(n) < c(n) \] and \( \check{q}(\gamma) \models \check{x}(n) \leq \check{x}^+(n) \). Therefore \( (p_1, q_1) \models \mathbb{P} \mathbb{B}_{\gamma+1} \check{x}(n) < \check{x}^+(n) \).

**Lemma □**

(4) To prove (4) we shall show \( V(\mathbb{B}, \check{\gamma}) = \models \square(\mathbb{L} \mathbb{O} \mathbb{C}, \omega^\omega, \mathbb{B}) \) where \( \mathbb{L} \mathbb{O} \mathbb{C} = \{ \phi : \phi \) is a function from \( \omega \) to \( \omega^\omega \) such that \( \exists k \in \omega \) \( |\phi(n)| \leq n^k \) for \( n \in \omega \} \) and \( \phi \models x \) if \( \forall \mathbb{V} \in \omega^\omega \mathbb{V}(\phi(n) \models x(n)) \) for \( \phi \in \mathbb{L} \mathbb{O} \mathbb{C} \) and \( x \in \omega^\omega \). Without loss of generality we can assume \( \mathbb{B} = \mathbb{D} \) is a complete Boolean algebra with strictly positive finitely additive measure \( \mu \) [1, p319 Lemma 6.5.18]. So it suffices to show the following lemma:

**Lemma 3.8.** Suppose \( \gamma \) is an ordinal and \( \mathbb{P} \) is a forcing notion which has a \( \mathbb{P} \)-name \( c \) such that for all \( \phi \in \mathbb{L} \mathbb{O} \mathbb{C} \cap V \models \exists n (\phi(n) \neq c(n)) \). Let \( \mathbb{B}_\gamma \) be a \( \gamma \)-stage finite support iteration of complete Boolean algebras with strictly additive measure \( \mu \) for each \( \gamma \). Let \( \phi \) be a \( \mathbb{B}_\gamma \)-name such that \( \models \mathbb{B}_\gamma \phi \in \mathbb{L} \mathbb{O} \mathbb{C} \). Then \( \models \mathbb{P} \mathbb{B}_\gamma \phi \neq \mathbb{C} \).

**Proof.** We proceed by induction on \( \gamma \). We shall prove only the successor step.

**Successor step:**
Suppose for \( \gamma \) the lemma holds. Let \( \hat{\phi} \) be a \( \mathbb{B}_{\gamma+1} \)-name such that \( \models \mathbb{B}_{\gamma+1} \hat{\phi} \in \mathbb{L} \mathbb{O} \mathbb{C} \). Let \( \psi \) (\( i < \omega \)), \( \check{p} \) (\( i < \omega \)) and \( \check{\psi} \) (\( i < \omega \)) be \( \mathbb{B}_\gamma \)-names such that
• $\models_B \psi_i \in \mathbb{LOC}$, $\hat{p}_i \in B$ and $\hat{k}_i \in \omega$ for $i < \omega$,

• $\models_B \psi_i \models_B \forall n \in \omega \left( \hat{\phi}_i(n) \leq n^{\hat{k}_i} \right)$" and

• $\models_B, \psi_i(n) = \{ j : \mu \left( [j \in \hat{\phi}(n)]_B \wedge \hat{p}_i \right) \geq \frac{1}{n} \}$.

Claim 3.8.1. $\models_B, \psi_i(n) \leq n^{k_{i+1}}$.

Let $m \in \omega$ and $(p_0, \hat{q}_0) \in \mathbb{P} \ast \mathcal{B}_{i+1}$. Without loss of generality we can find $i \in \omega$ and $n_i \in \omega$ such that $(p, \hat{q} \upharpoonright \gamma) \models_{\mathbb{P} \ast \mathcal{B}_i} \mu (\hat{q}(\gamma) \wedge \hat{p}_i) \geq \frac{1}{n_i}$. By induction hypothesis there exist $(p', \hat{q}') \leq_{\mathbb{P} \ast \mathcal{B}_i} (p, \hat{q} \upharpoonright \gamma)$ and $n \geq n_i, m$ such that $(p', \hat{q}') \models_{\mathbb{P} \ast \mathcal{B}_i} \hat{c}(n) \notin \psi_i(n)$. Without loss of generality we can assume $p'$ decides $\hat{c}(n)$ and $p' \models_B \hat{c}(n) = \iota$ for some $\iota \in \omega$. Since $(p', \hat{q}') \models_{\mathbb{P} \ast \mathcal{B}_i} l \notin \psi_i(n)$, $(p', \hat{q}') \models_{\mathbb{P} \ast \mathcal{B}_i} \mu \left( [l \in \hat{\phi}(n)]_B \wedge \hat{p}_i \wedge \hat{q}(\gamma) \right) > 0$.

Put $(p_1, \hat{q}_1) \in \mathbb{P} \ast \mathcal{B}_i$ so that $(p_1, \hat{q}_1 \upharpoonright \gamma) = (p', \hat{q}')$ and $(p_1, \hat{q}_1 \upharpoonright \gamma) \models_{\mathbb{P} \ast \mathcal{B}_i} \hat{q}_1(\gamma) = [l \notin \hat{\phi}(n)]_B \wedge \hat{p}_i \wedge \hat{q}(\gamma)$. Then $(p_1, \hat{q}_1) \models_{\mathbb{P} \ast \mathcal{B}_{i+1}} \hat{c}(n) = \iota \notin \hat{\phi}(n)$.

Lemma □

So We have $V^{(\mathbb{B} \ast \mathcal{D})} = \models (L \mathbb{OC}, \omega^\omega, \mathcal{Z})$.

Let $\{ C_{i,j} \}$ be a family of independent open sets with $\mu(C_{i,j}) = \frac{1}{(i+1)^2}$ for all $i, j$. Let $\Phi : \omega^\omega \rightarrow \mathcal{N}$ be the function such that

$$\Phi(f) = \bigcup_{n \geq n} C_{i,f(i)}.$$

For each $B \in \mathcal{N}$ fix a compact set $K_B \subset \omega^\omega \setminus B$ with $\mu(K_B \cap U) > 0$ for any open set $U$ with $K_B \cap U \neq \emptyset$. Let $\{ \sigma_{n}^B : n \in \omega \}$ list all $\sigma \in \omega^{<\omega}$ with $K_B \cap [\sigma] \neq \emptyset$. Put

$$g(B, n, i) = \{ j : K_B \cap [\sigma_{n}^B] \cap C_{i,j} = \emptyset \}$$

for $i, n \in \omega$. Fix $k(B, n)$ such that

$$|g(B, n, i)| \leq \frac{(i+1)^2}{2n+1}$$

for $i \geq k(B, n)$. Define $\Psi : \mathcal{N} \rightarrow \mathbb{LOC}$ by

$$\Psi(B)(i) = \bigcup_{k(B, n) \leq i} g(B, n, i).$$

Then $\Psi$ and $\Phi$ witness $(\mathcal{N}, \mathcal{N}, \mathcal{Z}) \leq_B (\mathbb{LOC}, \omega^\omega, \mathcal{Z})$ (see [1, Theorem 2.3.9]). So $V^{(\mathbb{B} \ast \mathcal{D})} = \models (\mathcal{N}, \mathcal{N}, \mathcal{Z})$.

Theorem □
Corollary 3.9. Each of the following are relatively consistent with ZFC:

(i) \( \kappa = \text{add}(\mathcal{M}) = \omega_2 + \lozenge(\text{cov}(\mathcal{N})) \) (see Diagram 1).

(ii) \( \kappa = \text{cov}(\mathcal{N}) = \text{cov}(\mathcal{M}) = \omega_2 + \lozenge(\text{b}) \) (see Diagram 2).

(iii) \( \kappa = \text{non}(\mathcal{M}) = \text{cov}(\mathcal{M}) = \omega_2 + \lozenge(\text{b}) + \lozenge(\text{cov}(\mathcal{N})) \) (see Diagram 3).

(iv) \( \kappa = \text{cov}(\mathcal{N}) = \text{add}(\mathcal{M}) = \omega_2 + \lozenge(\text{add}(\mathcal{N})) \) (see Diagram 4).

Proof. (i) Suppose \( V \models \text{CH} \). By Theorem 3.4 (1) \( V^{\mathbb{D}_{\omega_2}} \models \lozenge(\text{cov}(\mathcal{N})) \). Since \( \mathbb{D}_{\omega_2} \) adds \( \omega_2 \)-many dominating reals and Cohen reals, \( V^{\mathbb{D}_{\omega_2}} \models \kappa = \text{b} = \text{cov}(\mathcal{M}) = \omega_2 \). Since \( \text{add}(\mathcal{M}) = \min\{\text{b}, \text{cov}(\mathcal{M})\} \) (see [1], [8]),

\[ V^{\mathbb{D}_{\omega_2}} \models \lozenge(\text{cov}(\mathcal{N})) + \kappa = \text{add}(\mathcal{M}) = \omega_2. \]

Cichoń’s diagram for parametrized diamond looks as follows where an \( \omega_2 \) means the corresponding evaluation of the Borel invariant is \( \omega_2 \) while the parametrized diamond principle for the others hold.

Diagram 1.

(ii) Suppose \( V \models \text{CH} \). By Theorem 3.4 (2) \( V^{\mathbb{B}_{\omega_2}} \models \lozenge(\text{b}) \). Since \( \mathbb{B}_{\omega_2} \) adds \( \omega_2 \) many Cohen and random reals, \( V^{\mathbb{B}_{\omega_2}} \models \kappa = \text{cov}(\mathcal{N}) = \text{cov}(\mathcal{M}) = \omega_2 \). Hence

\[ V^{\mathbb{B}_{\omega_2}} \models \lozenge(\text{b}) + \kappa = \text{cov}(\mathcal{N}) = \text{cov}(\mathcal{M}) = \omega_2. \]

Diagram 2.

(iii) Suppose \( V \models \text{CH} \). By Theorem 3.4 (3) \( V^{\mathbb{E}_{\omega_2}} \models \lozenge(\text{cov}(\mathcal{N})) + \lozenge(\text{b}) \). Since \( \mathbb{E}_{\omega_2} \) adds \( \omega_2 \) many Cohen and eventually different reals, \( \kappa = \text{non}(\mathcal{M}) = \text{cov}(\mathcal{M}) = \omega_2 \). Hence

\[ V^{\mathbb{E}_{\omega_2}} \models \lozenge(\text{cov}(\mathcal{N})) + \lozenge(\text{b}) + \kappa = \text{non}(\mathcal{M}) = \text{cov}(\mathcal{M}). \]
(iv) Suppose $V \models CH$. By Theorem 3.4 (4) $V^{(\mathcal{B}+\mathcal{D})^\omega_2} \models \Diamond(\text{add}(\mathcal{N}))$. Since $(\mathcal{B}+\mathcal{D})^\omega_2$ adds $\omega_2$ many random, Cohen and dominating reals, $c = \text{cov}(\mathcal{N}) = \text{add}(\mathcal{M}) = \min\{b, \text{cov}(\mathcal{M})\} = \omega_2$. Hence

$$V^{(\mathcal{B}+\mathcal{D})^\omega_2} \models \Diamond(\text{add}(\mathcal{N})) + c = \text{cov}(\mathcal{N}) = \text{add}(\mathcal{M}) = \omega_2.$$ 

Hrušák asked the following question after a talk I gave at the 33rd Winter School on Abstract Analysis -Section of Topology held in the Czech Republic (2005 January).

**Question 3.10** (Hrušák). Let $\mathcal{B}$ be a amoeba forcing. Then $V^{h_{\omega_2}} \models \Diamond(s)$?

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**References**


