Independence number for partitions of ω

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Motivation

- The combinatorial structure of $\left(\wp(\omega)/fin, \leq_{fin}\right)$ is described by cardinal invariants of the continuum.

- We can define analogous cardinal invariants describing properties of $(\wp(\omega)/\mathcal{I}, \leq_{\mathcal{I}})$, $(Dense(\mathbb{Q})/nwd \leq_{nwd})$, $((\omega)^{\omega}, \leq^{*})$ or $((\omega)^{\omega}, {}^{*}\geq)$.

Program: Compare these cardinal invariants to investigate the similarities and differences between these structures.

In this talk, we focus on the independence number on $\left(\wp(\omega)/fin, \leq_{fin}\right)$ and $((\omega)^{\omega}, \leq^{*})$.

$\boxed{(\omega)}$

Definition 1.

$X \subset \wp(\omega)$ is a **partition** *of ω if $\forall x, y \in X(x \neq y \to x \cap y = \emptyset)$ and $\bigcup X = \omega$.*

$$(\omega) = \{X \subset \wp(\omega) : X \text{ is a partition of } \omega\}.$$

For $X, Y \in (\omega)$ X is **coarser** *than Y ($X \leq Y$) if*

$$\forall x \in X \exists Y' \subset Y (x = \bigcup Y').$$

Proposition 2. *$((\omega), \leq)$ is a lattice.*

$X \wedge Y =$ **infinimum** of X and Y.

$(\omega)^\omega$

$$\begin{aligned} (\omega)^\omega &= \{X \in (\omega) : |X| = \aleph_0\} \\ (\omega)^{<\omega} &= \{X \in (\omega) : |X| < \aleph_0\}. \end{aligned}$$

Definition 3. *Let $X, Y \in (\omega)^\omega$.*

X *is* **almost coarser** *than* Y *(*$X \leq^* Y$*) if* $\forall^\infty x \in X \exists Y'(x = \bigcup Y')$*.*

X *is* **non-trivial** *if* $\{\{n\} : n \in \omega\} \not\leq^* X$*.*

X *and* Y *are* **compatible** *(*$X \| Y$*) if* $X \wedge Y \in (\omega)^\omega$*.*

X *and* Y *are* **incompatible** *(*$X \perp Y$*) if* $X \wedge Y \in (\omega)^{<\omega}$

properties of $((\omega), \leq^*)$

As $\left(\wp(\omega)/fin, \leq_{fin}\right)$, $((\omega)^\omega, \leq^*)$ has the following properties:

Lemma 4. *(J.Cichoń, A.Krawczyk, B.Majcher-Iwanow, B.Węglorz)* *Suppose that $X_0 \geq X_1 \geq X_2 \geq \ldots$ is a decreasing sequence of $(\omega)^\omega$. Then there exists $Y \in (\omega)^\omega$ such that $Y \leq^* X_n$ for $n \in \omega$.*

Lemma 5. *(J.Cichoń, A.Krawczyk, B.Majcher-Iwanow,B.Węglorz)* *For $X, Y \in (\omega)^\omega$ if $X \not\leq^* Y$, then there exists $Z \in (\omega)^\omega$ such that $Z \leq^* X$ and $Z \perp Y$.*

dual-independent family

$((\omega)^\omega, \leq^*)$ doesn't have any natural complementation.

Definition 6. *Let $\mathcal{I}$ be a subset of $(\omega)^\omega$. $\mathcal{I}$ is a dual-independent family if for all $\mathcal{A}$ and $\mathcal{B}$ finite subsets of $\mathcal{I}$ with $\mathcal{A} \cap \mathcal{B} = \emptyset$ there exists $C \in (\omega)^\omega$ such that*

(i) $C \leq^* A$ for $A \in \mathcal{A}$ and

(ii) $C \perp B$ for $B \in \mathcal{B}$.

Then define dual-independence number $\mathfrak{i}_d$ by

$$\mathfrak{i}_d = \min\{|\mathcal{I}| : \mathcal{I} \text{ is a maximal dual-independent family}\}.$$

$\mathfrak{i}_d$ and other cardinals

Definition 7. $\mathcal{R} \subset (\omega)^\omega$ *is a dual-reaping family if* $\forall X \in (\omega)^\omega \exists Y \in \mathcal{R}(X \perp Y$ *or* $Y \leq^* X)$.

$$\mathfrak{r}_d = \min\{|\mathcal{R}| : \mathcal{R} \text{ is a dual-reaping family}\}.$$

As $\mathfrak{r} \leq \mathfrak{i}$, we have the following:

Proposition 8. $\mathfrak{r}_d \leq \mathfrak{i}_d$.

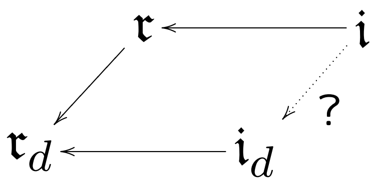
$\mathfrak{r}_d$ and $\mathfrak{r}$ satisfies the following inequality:

Proposition 9. *(J.Cichoń, A.Krawczyk, B.Majcher-Iwanow,B.Węglorz)*

$$\mathfrak{r}_d \leq \mathfrak{r}.$$

Open Problem. $\mathfrak{i}_d \leq \mathfrak{i}$ or it is consistent that $\mathfrak{i} < \mathfrak{i}_d$?

independence number and continuum.



$\kappa \rightarrow \lambda$ means that $\lambda \leq \kappa$ is provable in ZFC.

Theorem 10. *(J.Cichoń, A.Krawczyk, B.Majcher-Iwanow, B.Węglorz)* *MA implies $\mathfrak{r}_d = \mathfrak{c}$. Therefore MA implies $\mathfrak{i}_d = \mathfrak{c}$.*

Question 1. *Con($\mathfrak{i}_d < \mathfrak{c}$)?*

Lemma 11. *(J.Cichoń, A.Krawczyk, B.Majcher-Iwanow, B.Węglorz) If $X, Y \in (\omega)^\omega$ and $X \not\leq^* Y$, then there exists an infinite sequence $\{a_n\}_{n\in\omega}$ of different elements of X such that*

$$\forall n \in \omega \exists y \in Y \left(y \cap a_{2n} \neq \emptyset \wedge y \cap a_{2n+1} \neq \emptyset\right)$$

or there exists a finite subset A of X such that the set

$$\{x \in X \setminus A : \exists y \in Y (x \cap y \neq \emptyset \wedge \bigcup A \cap y \neq \emptyset)\}$$

is infinite.

Lemma 12. *(Minami) If $X \in (\omega)^\omega$ and $\mathcal{B}$ is a finite subset of $(\omega)^\omega$ such that $X \not\leq^* B$ for $B \in \mathcal{B}$, then there exists $Z \leq X$ such that $Z \perp B$ for $B \in \mathcal{B}$.*

This statement doesn't hold for $\left(\wp(\omega)/fin, \leq_{fin}\right)$.

Example 2. *Let $B_1 \in [\omega]^\omega$ and $B_2 \in [\omega]^\omega$ with $B_1 \cap B_2 = \emptyset$. Put $X \subset B_1 \cup B_2$ with $B_i \cap X \in [\omega]^\omega$ for $i = 1, 2$.*

Then there is no infinite $Z \subset X$ with $Z \cap B_i = \emptyset$ for $i = 1, 2$.

Corollary 13. *$\mathcal{I}$ is dual-independent if and only if for each finite subset $\mathcal{A}$ of $\mathcal{I}$ and $B \in \mathcal{I} \setminus \mathcal{A}$*

$$\bigwedge \mathcal{A} \not\leq^* B.$$

$\Big(\wp(\omega)/fin, \leq_{fin}\Big)$ doesn't have this properties.

Example 3. *Pick $X, Y \in [\omega]^\omega$ so that $X \cap Y \in [\omega]^\omega$, $X \setminus Y \in [\omega]^\omega$, $Y \setminus X \in [\omega]^\omega$ and $X \cup Y = \omega$. Put $\mathcal{I} = \{X, Y\}$.*

$\mathcal{I}$ satisfies that for any finite $\mathcal{A} \subset \mathcal{I}$ and $B \in \mathcal{I} \setminus \mathcal{A}$, $\bigcap \mathcal{A} \not\subset^* B$, but $\mathcal{I}$ is not independent family. Because

$$(\omega \setminus X) \cap (\omega \setminus Y) = \emptyset.$$

Cohen forcing and $\mathfrak{i}_d$

Theorem 14. *If $V \models CH$, then $V^{\mathbb{C}_{\omega_2}} \models \mathfrak{i}_d < \mathfrak{c}$.*

To prove Theorem 14 we use the following lemma.

Lemma 15. *Assume $p \in \mathbb{C}$, $\mathcal{I}$ is a countable dual-independent family and $\dot{X}$ is a $\mathbb{C}$-name such that $p \Vdash$ "$\dot{X}$ is a non-trivial infinite partition of ω and $\{\dot{X}\} \cup \mathcal{I}$ is dual-independent". Then there exists $X^* \in (\omega)^\omega \cap V$ such that $\{X^*\} \cup \mathcal{I}$ is dual-independent and $p \Vdash \dot{X} \perp X^*$.*

Corollary 16. *$Con(\mathfrak{i}_d < \mathfrak{r})$.*