Around splitting and reaping number for partitions of ω

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Motivation

- The combinatorial structure of $\left(\wp(\omega)/fin, \leq_{fin}\right)$ is described by cardinal invariants of the continuum.

- We can define analogous cardinal invariants describing properties of $(\wp(\omega)/\mathcal{I}, \leq_{\mathcal{I}})$, $(Dense(\mathbb{Q})/nwd \leq_{nwd})$, $((\omega)^\omega, \leq^*)$ or $((\omega)^\omega, {}^* \geq)$.

Program: Compare these cardinal invariants to investigate the similarities and differences between these structures.

In this talk, we focus on the reaping number and splitting number on $\left(\wp(\omega)/fin, \leq_{fin}\right)$ and $((\omega)^\omega, \leq^*)$.

$\boxed{(\omega)}$

Definition 1.

$X \subset \wp(\omega)$ is a **partition** *of ω if $\forall x, y \in X (x \neq y \to x \cap y = \emptyset)$ and $\bigcup X = \omega$.*

$$(\omega) = \{X \subset \wp(\omega) : X \text{ is a partition of } \omega\}.$$

For $X, Y \in (\omega)$ X is **coarser** *than Y ($X \leq Y$) if*

$$\forall x \in X \exists Y' \subset Y (x = \bigcup Y').$$

Proposition 2. *$((\omega), \leq)$ is a lattice.*

$$X \wedge Y = \textbf{infimum} \text{ of } X \text{ and } Y.$$

$\boxed{(\omega)^\omega}$

$$\begin{aligned} (\omega)^\omega &= \{X \in (\omega) : |X| = \aleph_0\} \\ (\omega)^{<\omega} &= \{X \in (\omega) : |X| < \aleph_0\}. \end{aligned}$$

Definition 3. *Let* $X, Y \in (\omega)^\omega$.

X *is* **almost coarser** *than* Y *(*$X \leq^* Y$*) if* $\forall^\infty x \in X \exists Y' \subset Y (x = \bigcup Y')$.

X *and* Y *are* **compatible** *(*$X \| Y$*) if* $X \wedge Y \in (\omega)^\omega$.

X *and* Y *are* **incompatible** *(*$X \perp Y$*) if* $X \wedge Y \in (\omega)^{<\omega}$

splitting family and reaping family

Definition 4.

For $X, Y \in (\omega)^\omega$ X **dual-splits** Y *if* $X \wedge Y \in (\omega)^\omega$ *and* $Y \not\leq^* X$.

$\mathcal{S} \subset (\omega)^\omega$ *is a* **dual-splitting family** *if* $\forall Y \in (\omega)^\omega \exists X \in \mathcal{S}(X$ *dual-splits* $Y)$.

$$\mathfrak{s}_d = \min\{|\mathcal{S}| : \mathcal{S} \text{ is a dual-splitting family}\}.$$

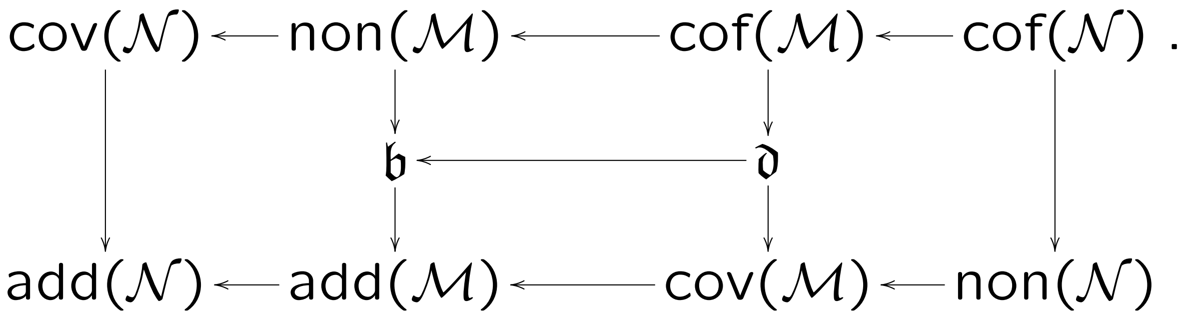
$\mathcal{R} \subset (\omega)^\omega$ *is a* **dual-reaping family** *if*

$$\forall Y \in (\omega)^\omega \exists X \in \mathcal{R}\ (Y \text{ doesn't dual-split } X).$$

$$\mathfrak{r}_d = \min\{|\mathcal{R}| : \mathcal{R} \text{ is a dual-reaping family}\}.$$

Cichoń's diagram

Let $\mathcal{M}$ be the meager ideal. Let $\mathcal{N}$ be the null ideal. Then the following relation holds.



Cichoń's diagram

$\kappa \to \lambda$ means $\kappa \geq \lambda$ is provable in ZFC.

$\mathfrak{r}_d$, $\mathfrak{s}_d$ and cardinal invariants in Cichoń's diagram

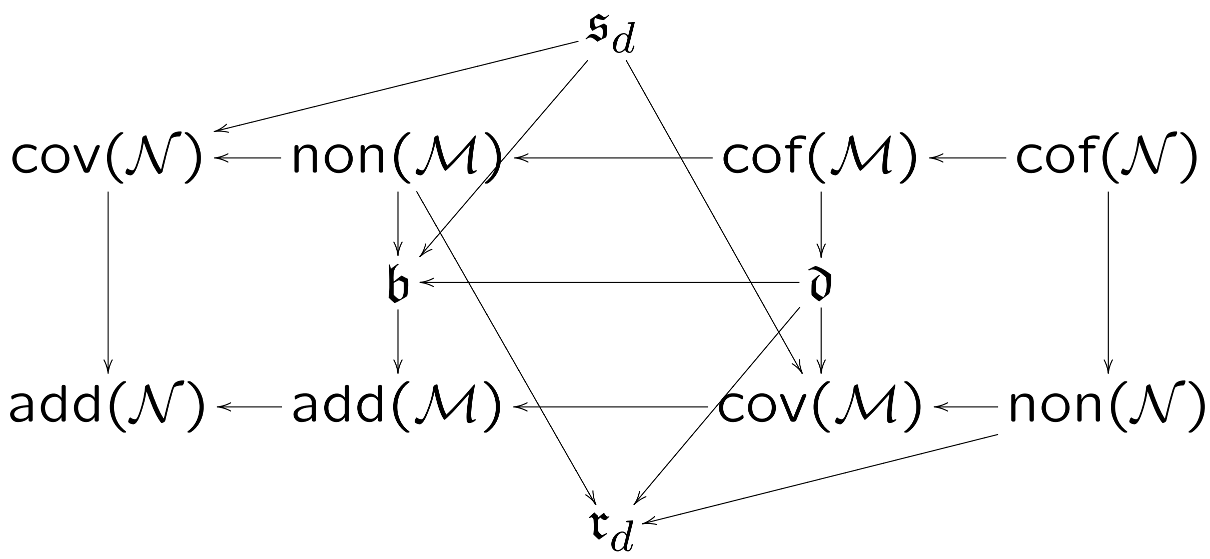
Theorem 5.
(i) (Cichoń, Krawczyk, Majcher-Iwanow, Węglorz)

$$\mathfrak{s}_d \geq cov(\mathcal{M}).$$

(ii) $\mathfrak{s}_d \geq cov(\mathcal{N})$. $\mathfrak{r}_d \leq non(\mathcal{M}), non(\mathcal{N})$.

(iii) (Kamo) $\mathfrak{r}_d \leq \mathfrak{d}$. $\mathfrak{s}_d \geq \mathfrak{b}$.

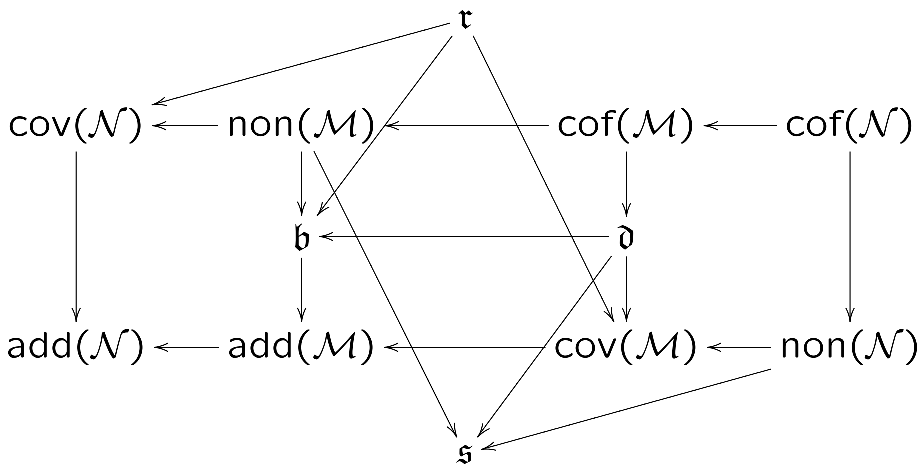
$\mathfrak{r}_d$, $\mathfrak{s}_d$ and Cichoń's diagram



$\kappa \to \lambda$ means $\kappa \geq \lambda$ is provable in ZFC.

$\mathfrak{r}$, $\mathfrak{s}$ and Cichoń's diagram

Let $\mathfrak{r}$ and $\mathfrak{s}$ be the reaping number and splitting number for $\left(\wp(\omega)/fin, \leq_{fin}\right)$ respectively.



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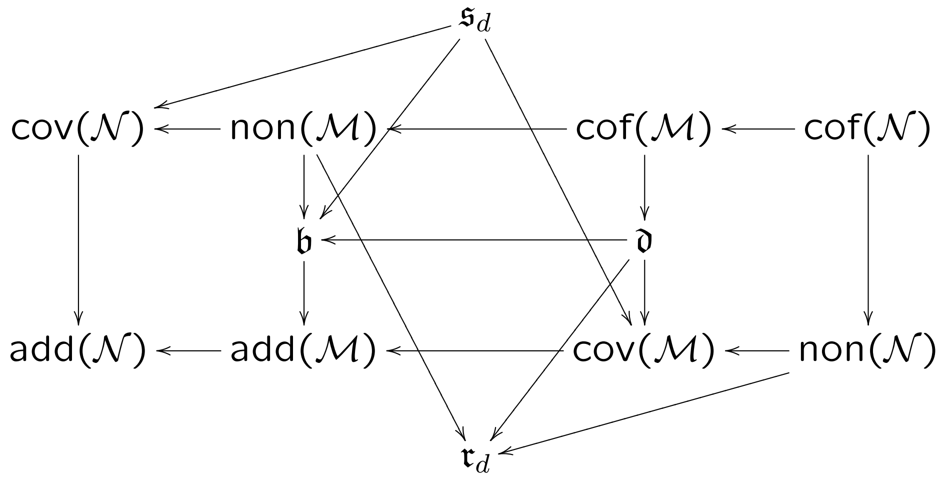

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$\mathfrak{r}$, $\mathfrak{s}$, $\mathfrak{r}_d$, $\mathfrak{s}_d$ and Cichoń's diagram

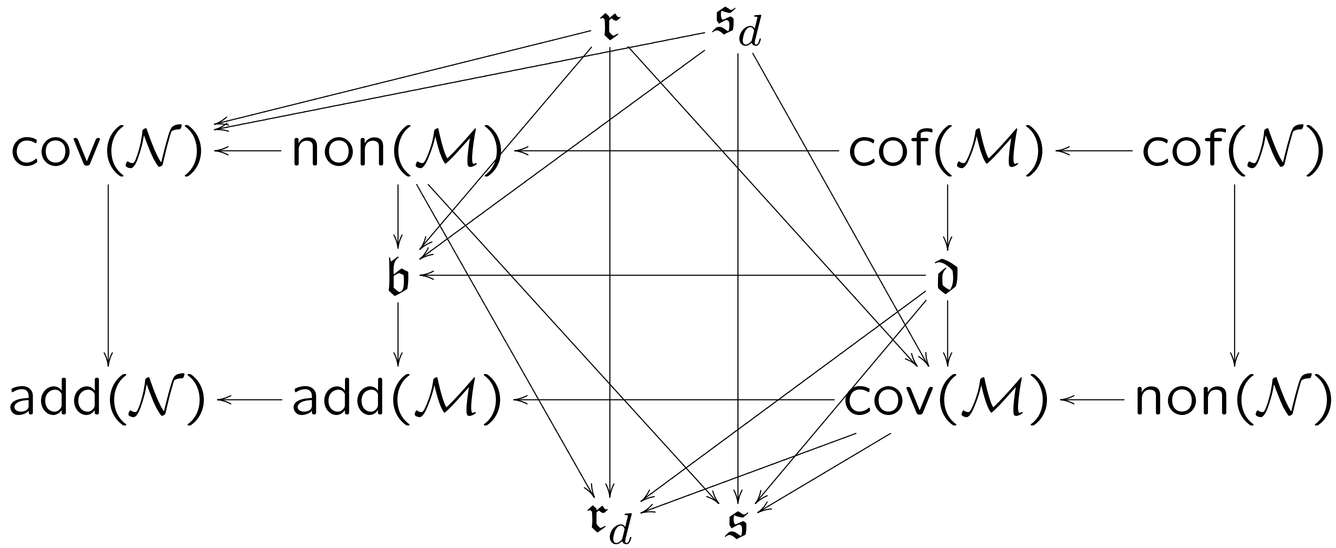
Theorem 6.

(1) (Cichoń, Krawczyk, Majcher-Iwanow, Węglorz)

$$\mathfrak{r}_d \leq \mathfrak{r}.$$

(2) (Kamburelis and Węglorz)

$$\mathfrak{s}_d \geq \mathfrak{s}.$$



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$\mathfrak{s}$, $\mathfrak{r}$, $\mathfrak{s}_d$ and $\mathfrak{r}_d$

Theorem 7 (Blass and Shelah)**.** $Con(\mathfrak{u} < \mathfrak{s})$.

So $Con(\mathfrak{r}_d < \mathfrak{s})$. $Con(\mathfrak{r} < \mathfrak{s}_d)$.

Let $\mathbb{DS}$ be a c.c.c forcing notion introduced by Cichoń, Krawczyk, Majcher-Iwanow and Węglorz.

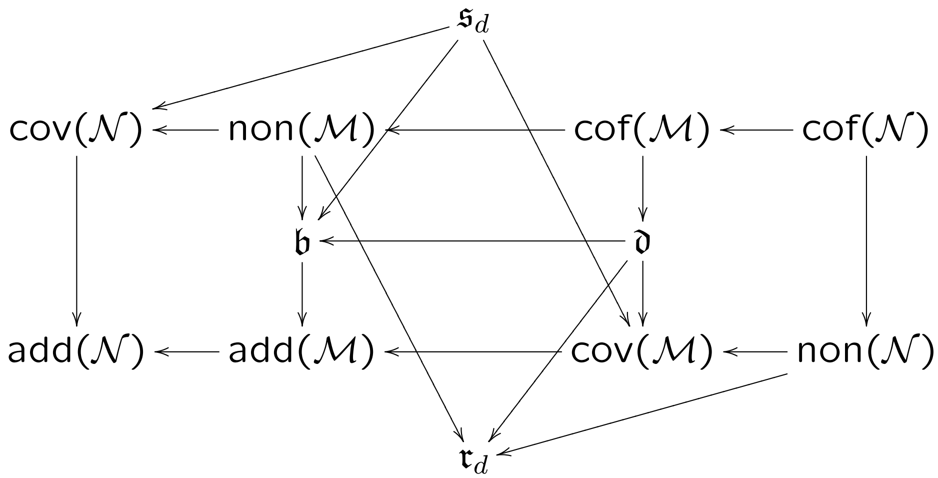
Theorem 8. *(Cichoń, Krawczyk, Majcher-Iwanow and Węglorz) If* $V \models CH$*, then* $V^{\mathbb{DS}_{\omega_2}} \models \mathfrak{r}_d = \mathfrak{c}$.

Since $\mathbb{DS}$ is a Suslin c.c.c, $V^{\mathbb{DS}_{\omega_2}} \models \mathfrak{s} = \omega_1$. Hence $Con(\mathfrak{s} < \mathfrak{r}_{\mathfrak{d}})$.

Theorem 9. *(Minami) If* $V \models MA$*, then* $V^{\mathbb{DS}_{\omega 1}} \models \mathfrak{r} > \mathfrak{s}_d$*. So* $Con(\mathfrak{r} > \mathfrak{s}_d)$*.*

Question 9.1. $Con(\mathfrak{s}_d < \mathfrak{r}_d)$ *?*

$\mathfrak{r}_d$, $\mathfrak{s}_d$ and Cichoń's diagram 2



Theorem 10 (Brendle)**.**
If $V \models CH$*, then* $V^{\mathbb{DS}_{\omega_2}} \models \mathfrak{r}_d > \mathfrak{b}$*.*
If $V \models MA$*, then* $V^{\mathbb{DS}_{\omega_1}} \models \mathfrak{s}_d < \mathfrak{d}$*.*

Question 10.1. $\mathfrak{s}_d \leq$ *cof*($\mathcal{M}$)*?* $\mathfrak{r}_d \geq$ *add*($\mathcal{M}$)*?*

Hechler forcing and $\mathfrak{r}_d$ and $\mathfrak{s}_d$

By $\mathbb{D}$ we denote the Hechler forcing.

Theorem 11. *(Minami) Let $\mathbb{D}_\alpha$ be the α-stage finite support iteration of Hechler forcing.*

If $V \models CH$, then $V^{\mathbb{D}_{\omega_2}} \models \mathfrak{r}_d = \omega_1 < add(\mathcal{M}) = \omega_2$.

If $V \models MA$, then $V^{\mathbb{D}_{\omega_1}} \models \mathfrak{s}_d = \omega_2 > cof(\mathcal{M}) = \omega_1$.

So $Con(\mathfrak{r}_d < add(\mathcal{M}))$ and $Con(\mathfrak{s}_d > cof(\mathcal{M}))$.