Conjectures of Rado and Chang, and Special Aronszajn Trees

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Chang’s Conjecture and Aronszajn trees
Some applications of RC

Special Aronszajn trees of larger heights

Rado’s Conjecture and Weak Squares
Principal Definitions

Definition (Rado’s Conjecture in Todorčević’s equivalent version, 1983)

A tree $T$ of height $\omega_1$ is the union of countably many antichains (special) if and only if every subtree of $T$ of size $\aleph_1$ is special.
Theorem (Todorčević, 1993)

*Rado’s Conjecture implies (some examples):*

1. $\theta = \mathbb{N}$ for all regular $\theta \geq \aleph_2$,
2. the Singular Cardinal Hypothesis,
3. $2^{\aleph_0} \leq \omega^2$,
4. $\square_\kappa$ fails for every uncountable cardinal $\kappa$,
5. $CC^*$. 

Theorem (Feng, 1999)

Rado’s Conjecture implies the presaturation of the nonstationary ideal on $\omega_1$. 

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*Rado’s Conjecture implies the presaturation of the nonstationary ideal on $\omega_1$.***
We consider the following strong version of Chang's Conjecture, which we denote by $CC^*$:

**Definition ($CC^*$)**

There are arbitrarily large uncountable regular cardinals $\theta$ such that for every well-ordering $\langle H_\theta, \in, < \rangle$ and every countable elementary submodel $M \prec \langle H_\theta, \in, < \rangle$, and every ordinal $\alpha < \omega^2$, there exists an elementary countable submodel $M^* \prec \langle H_\theta, \in, < \rangle$ such that

1. $M^* \supseteq M$.
2. $M \setminus \omega_1 = M^* \setminus \omega_1$.
3. $A_{M^*} = M^* \setminus \omega_2 \setminus M \setminus \omega_2 \neq \emptyset$ and $\min(A_{M^*}) \geq \alpha$. 

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2. There is a special $\aleph_2$-Aronszajn tree.
Proposition

No special $\aleph_2$-Aronszajn tree admits an ascending $\omega_2$-path of countable subsets, i.e., a sequence $A_\xi \in [T]^{\aleph_0}(\xi < \omega_2)$ of pairwise disjoint sets such that for every $\xi < \eta$, the height of every node of $A_\xi$ is smaller than the height of every node of $A_\eta$ and
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$$\exists x \in A_\xi \exists y \in A_\eta \ x <_T y.$$
Let $c : T \to \omega_1$ be a specializing map, i.e. for every $x, y \in T$ with $x <_T y$, $c(x) \neq c(y)$. In order to obtain a contradiction, suppose $T$ admits an ascending $\omega_2$-path of countable subsets $\langle A_\xi \rangle_{\xi \in \omega_2}$. For $\xi < \omega_2$, let $\sup c'' A_\xi = \alpha_\xi$. Since $|A_\xi| = \aleph_0$, $\alpha_\xi < \omega_1$. The map $\xi \mapsto \alpha_\xi$ gives a partition of $\omega_2$ into $\omega_1$ parts. By regularity of $\omega_2$, there is $\Gamma \subseteq \omega_2$ of cardinality $\omega_2$, and $\alpha \in \omega_1$ such that for every $\xi \in \Gamma$, $\alpha_\xi = \alpha$. Let $\delta$ the $\omega_1$-th element of $\Gamma$. In particular, $|\Gamma \setminus \delta| = \aleph_1$. 

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$$\sup c'' A_\xi = \alpha_\xi.$$

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By the definition of ascending path, for every $\xi \in \Gamma \cap \delta$, there are $y_\xi \in A_\delta$ and $x \in A_\xi$ with $x < T y_\xi$. 
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Also remark that $\text{pred}(y) \cap A_\xi \neq \emptyset$ for every $\xi$ in $\Sigma$, since $x_\xi \in \text{pred}(y) \cap A_\xi$. So for $\xi < \eta$ in $\Sigma$, since $y \in A_\delta$, and $x_\eta, x_\xi <_T y$, we have $x_\xi <_T x_\eta$ (since $T$ is a tree).
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and $\alpha$ is countable, contradiction.
From now on we fix a tree $T$ of height $\omega_2$ and levels of cardinality $\aleph_1$. We assume that the domain of $T$ is equal to $\omega_2$. Let $e : \omega_2 \times \omega_1 \to T$ be a bijective function such that for every $\delta \in \omega_2$ and for every $\xi \in \omega_1$, $e(\delta, \xi) \in T_\delta$, and $e(\delta, \xi) \geq \delta$. 
From now on we fix a tree $T$ of height $\omega_2$ and levels of cardinality $\aleph_1$. We assume that the domain of $T$ is equal to $\omega_2$. Let $e : \omega_2 \times \omega_1 \to T$ be a bijective function such that for every $\delta \in \omega_2$ and for every $\xi \in \omega_1$, $e(\delta, \xi) \in T_\delta$, and $e(\delta, \xi) \geq \delta$. Let $\theta$ be sufficiently large such that $T, e$ and all relevant parameters are members of $H_\theta$. 
Lemma

Assume $\text{CC}^*$ and that $T$ is a special $\aleph_2$-Aronszajn tree. For every $M \prec H_\theta$ countable, we can find $M_0, M_1 \prec H_\theta$ countable such that

1. $M \setminus \omega_1 = M_0 \setminus \omega_1 = M_1 \setminus \omega_1$,
2. $A_0 = M_0 \setminus \omega_2 \setminus M \setminus \omega_2 \neq \emptyset$, $A_1 = M_1 \setminus \omega_2 \setminus M \setminus \omega_2 \neq \emptyset$, and $\sup(A_0) < \min(A_1)$,
3. For every $x \in A_0 \setminus T$ and for every $y \in A_1 \setminus T$, $x \not\leq_T y$ and $y \not\leq_T x$. 
Lemma

Assume \(\text{CC}^*\) and that \(T\) is a special \(\aleph_2\)-Aronszajn tree. For every \(M \prec H_\theta\) countable, we can find \(M_0, M_1 \prec H_\theta\) countable such that

1. \(M \cap \omega_1 = M_0 \cap \omega_1 = M_1 \cap \omega_1\),
Lemma

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1. $M \cap \omega_1 = M_0 \cap \omega_1 = M_1 \cap \omega_1$,
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3. For every $x \in A_0 \cap T$ and for every $y \in A_1 \cap T$, $x \not\leq_T y$ and $y \not\leq_T x$. 

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Proof of Lemma

Otherwise, we could produce an $\omega_2$-ascending path of countable subsets of $T$, contradicting previous Proposition.
Lemma
Assume $\text{CC}^*$ and the negation of $\text{CH}$. If $T$ is a special $\aleph_2$-Aronszajn tree, the set

$$S_T = \{ A \in [\omega_2]^\omega : \forall x \in T (\text{pred}(x) \cap A \text{ is bounded in } \sup(A)) \}$$

is stationary.
Proof of Lemma

Let \( f : [\omega_2]^{<\omega} \rightarrow \omega_2 \). Using previous Lemma, build a binary tree \( \langle M_\sigma \rangle_{\sigma \in 2^{<\omega}} \) of countable elementary submodels of \( H_\theta \) within the set \( C_f \) of closure points of \( f \) (i.e. \( X \in C_f \) iff for every \( e \in [X]^{<\omega} \), \( f(e) \in X \)) with the property that for every \( \sigma \in 2^{<\omega} \)
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1. \( M_\sigma \cap \omega_1 = M_{\sigma \upharpoonright 0} \cap \omega_1 = M_{\sigma \upharpoonright 1} \cap \omega_1 \),
Proof of Lemma

Let \( f : [\omega_2]^{<\omega} \rightarrow \omega_2 \). Using previous Lemma, build a binary tree \( \langle M_\sigma \rangle_{\sigma \in 2^{<\omega}} \) of countable elementary submodels of \( H_\theta \) within the set \( C_f \) of closure points of \( f \) (i.e. \( X \in C_f \) iff for every \( e \in [X]^{<\omega} \), \( f(e) \in X \)) with the property that for every \( \sigma \in 2^{<\omega} \)

1. \( M_\sigma \cap \omega_1 = M_{\sigma \uparrow 0} \cap \omega_1 = M_{\sigma \uparrow 1} \cap \omega_1 \),
2. \( M_\sigma \cap \omega_2 \subsetneq M_{\sigma \uparrow 0} \cap \omega_2 \) and \( M_\sigma \cap \omega_2 \subsetneq M_{\sigma \uparrow 1} \cap \omega_2 \).
Proof of Lemma

Let \( f : [\omega_2]^{<\omega} \to \omega_2 \). Using previous Lemma, build a binary tree \( \langle M_\sigma \rangle_{\sigma \in 2^{<\omega}} \) of countable elementary submodels of \( H_\theta \) within the set \( C_f \) of closure points of \( f \) (i.e. \( X \in C_f \) iff for every \( e \in [X]^{<\omega} \), \( f(e) \in X \)) with the property that for every \( \sigma \in 2^{<\omega} \)

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2. \( M_\sigma \cap \omega_2 \subsetneq M_\sigma \downarrow_0 \cap \omega_2 \) and \( M_\sigma \cap \omega_2 \subsetneq M_\sigma \downarrow_1 \cap \omega_2 \),
3. For every \( x \in T \cap M_\sigma \downarrow_0 \setminus M_\sigma \) and for every \( y \in T \cap M_\sigma \downarrow_1 \setminus M_\sigma \), \( x \not\leq_T y \) and \( y \not\leq_T x \),
Let $f : [\omega_2]^{<\omega} \rightarrow \omega_2$. Using previous Lemma, build a binary tree $\langle M_\sigma \rangle_{\sigma \in 2^{<\omega}}$ of countable elementary submodels of $H_\theta$ within the set $C_f$ of closure points of $f$ (i.e. $X \in C_f$ iff for every $e \in [X]^{<\omega}$, $f(e) \in X$) with the property that for every $\sigma \in 2^{<\omega}$

1. $M_\sigma \cap \omega_1 = M_{\sigma \restriction 0} \cap \omega_1 = M_{\sigma \restriction 1} \cap \omega_1$,
2. $M_\sigma \cap \omega_2 \subsetneq M_{\sigma \restriction 0} \cap \omega_2$ and $M_\sigma \cap \omega_2 \subsetneq M_{\sigma \restriction 1} \cap \omega_2$,
3. For every $x \in T \cap M_{\sigma \restriction 0} \setminus M_\sigma$ and for every $y \in T \cap M_{\sigma \restriction 1} \setminus M_\sigma$, $x \not\leq_T y$ and $y \not\leq_T x$,
4. For every $r \in 2^\omega$, if $M_r = \bigcup_{n \in \omega} M_{r \upharpoonright n}$, then for every $r, r' \in 2^\omega$, $\sup(M_r \cap \omega_2) = \sup(M_{r'} \cap \omega_2)$. 
Proof of Lemma

Let $\delta$ be the common supremum of every $M_r \cap \omega_2$, $r \in 2^\omega$. 
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Claim

There is $r \in 2^\omega$ such that $M_r \cap \omega_2 \in S$. 
Proof.
Suppose not. Then for every $r \in 2^\omega$, there is $x_r \in T_\delta \cap M_r$ such that for every $\text{pred}(x_r) \cap M_r$ is unbounded in $\delta$.
Proof.
Suppose not. Then for every \( r \in 2^\omega \), there is \( x_r \in T_\delta \cap M_r \) such that for every \( \text{pred}(x_r) \cap M_r \) is unbounded in \( \delta \). But by the construction of the tree, for every \( r, r' \in 2^\omega \), \( r \neq r' \) implies \( x_r \neq x_{r'} \).
Proof.
Suppose not. Then for every $r \in 2^\omega$, there is $x_r \in T_\delta \cap M_r$ such that for every $\text{pred}(x_r) \cap M_r$ is unbounded in $\delta$. But by the construction of the tree, for every $r, r' \in 2^\omega$, $r \neq r'$ implies $x_r \neq x_{r'}$. Therefore, the application $r \mapsto x_r$ is an injection from $2^\omega$ to $T_\delta$. However, $|T_\delta| = \aleph_1$, contradicting the assumption that CH does not hold.
We are now ready to finish the proof of our Theorem. From the previous lemma we know that the set $S_T$ is stationary in $[\omega_2]^{\aleph_0}$. We need the following result of Todorcevic.
We are now ready to finish the proof of our Theorem. From the previous lemma we know that the set $S_T$ is stationary in $[\omega_2]^{\aleph_0}$. We need the following result of Todorcevic.

**Lemma**

$CC^*$ implies that for every stationary set $E \subseteq [\omega_2]^\omega$, there is an uncountable ordinal $\alpha \in \omega_2$ such that $E \cap [\alpha]^\omega$ is stationary.
It follows that there is \( \alpha \in \omega_2 \) such that \( S_T^\alpha = S'_T \cap [\alpha]^{\omega} \) is stationary, where \( S'_T \) is the intersection of \( S_T \) with the club of all countable subsets of \( \omega_2 \) closed under the level enumeration function \( e \) of \( T \). Pick an \( x \in T \) of height \( \geq \alpha \). Then \( \text{pred}(x) \cap \alpha \) is unbounded in \( \alpha \). From the definition of \( S_T \), we know that for every \( A \in S_T^\alpha \) there is \( h(A) \in A \) such that
It follows that there is $\alpha \in \omega_2$ such that $S^\alpha_T = S'_T \cap [\alpha]^{\omega}$ is stationary, where $S'_T$ is the intersection of $S_T$ with the club of all countable subsets of $\omega_2$ closed under the level enumeration function $e$ of $T$. Pick an $x \in T$ of height $\geq \alpha$. Then $\text{pred}(x) \cap \alpha$ is unbounded in $\alpha$. From the definition of $S_T$, we know that for every $A \in S^\alpha_T$ there is $h(A) \in A$ such that

$$h(A) \geq \sup(\text{pred}(x) \cap A).$$
It follows that there is $\alpha \in \omega_2$ such that $S^\alpha_T = S'_T \cap [\alpha]^\omega$ is stationary, where $S'_T$ is the intersection of $S_T$ with the club of all countable subsets of $\omega_2$ closed under the level enumeration function $e$ of $T$. Pick an $x \in T$ of height $\geq \alpha$. Then $\text{pred}(x) \cap \alpha$ is unbounded in $\alpha$. From the definition of $S_T$, we know that for every $A \in S^\alpha_T$ there is $h(A) \in A$ such that

$$h(A) \geq \sup(\text{pred}(x) \cap A).$$

Using the Pressing Down Lemma for stationary sets, we can find a stationary set $S \subseteq S^\alpha_T$ and $\xi < \alpha$ such that $h(A) = \xi$ for all $A \in S$. Since $S$, in particular, is a cofinal subset of $[\alpha]^\omega$, this means that $\text{pred}(x) \cap \alpha$ is bounded in $\alpha$, a contradiction.
We will just cite the following result that naturally supplements the result presented above.
We will just cite the following result that naturally supplements the result presented above.

**Theorem (Todorcevic, T. )**

*Rado’s Conjecture implies that there are no special $\kappa^+$-Aronszajn trees for any singular cardinal $\kappa$ of cofinality $\omega$.**
Reall that $\Box^*_\kappa$ is the statement that there is a sequence $\langle C_\alpha : \alpha < \kappa^+ \rangle$ such that $C_\alpha$ is a club in $\alpha$ of order type at most $\kappa$ and such that for all $\alpha < \kappa^+$,
Reall that $\Box^*_\kappa$ is the statement that there is a sequence $\langle C_\alpha : \alpha < \kappa^+ \rangle$ such that $C_\alpha$ is a club in $\alpha$ of order type at most $\kappa$ and such that for all $\alpha < \kappa^+$,

$$|\{C_\beta \cap \alpha : \alpha \leq \beta < \kappa^+\}| \leq \kappa.$$
Reall that $\square^*_\kappa$ is the statement that there is a sequence $\langle C_\alpha : \alpha < \kappa^+ \rangle$ such that $C_\alpha$ is a club in $\alpha$ of order type at most $\kappa$ and such that for all $\alpha < \kappa^+$,

$$|\{ C_\beta \cap \alpha : \alpha \leq \beta < \kappa^+ \}| \leq \kappa.$$

$\square^*_\kappa$ is a consequence of the cardinal assumption $\kappa^{<\kappa} = \kappa$ and, it is well-known, $\square^*_\kappa$ is equivalent to the existence of a special $\kappa^+$-Aronszajn tree.
Since RC implies $CC^*$ we get the following corollaries of Theorem 5 and 11.
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**Corollary (RC)**

□*ω1 and CH are equivalent.
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**Corollary (RC)**

$\Box^*_{\omega_1}$ and $CH$ are equivalent.

**Corollary**

$RC$ implies the negation of $\Box^*_\kappa$ for every singular cardinal of cofinality $\omega$. 
Recall the following variation on Jenen’s principle $\square_\kappa$ introduced by Schimmerling (1995): For cardinals $\lambda \leq \kappa$, let $\square_\kappa^\lambda$ be the statement that there is a sequence $\langle C_\alpha : \alpha < \kappa^+ \rangle$ such that:

1. $C_\alpha$ is a family of closed subsets of $\alpha$ with at least one unbounded in $\alpha$.
2. $|C_\alpha| \leq \lambda$ and $\text{otp}(C_\alpha) \leq \kappa$ for all $C_\alpha \in C_\alpha$.
3. If $C_\beta \in C_\beta$ and if $\alpha$ is a limit point of $C_\beta$, then $C_\beta \setminus C_\alpha \in C_\alpha$. 

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1. $C_\alpha$ is a family of closed subsets of $\alpha$ with at least one unbounded in $\alpha$.
2. $|C_\alpha| \leq \lambda$ and $\text{otp}(C) \leq \kappa$ for all $C \in C_\alpha$.
3. If $C \in C_\beta$ and if $\alpha$ is a limit point of $C$, then $C \cap \alpha \in C_\alpha$. 
Clearly, $\square^*_\kappa$ is equivalent to $\square^\kappa_\kappa$. The purpose of this section is to prove the following result:
Clearly, $\Box^*\kappa$ is equivalent to $\Box^\kappa\kappa$. The purpose of this section is to prove the following result:

**Theorem**

RC implies the negation of $\Box^\kappa\lambda$ for every $\kappa$ and $\lambda$ such that $\lambda < \text{cof} (\kappa)$. 
Fix a $\square^\lambda_\kappa$-sequence $\langle C_\alpha : \alpha < \kappa^+ \rangle$ and assume $\text{cof}(\kappa) > \lambda$. We shall show that $\mathcal{RC}$ fails. We consider the tree $T$ of all countable closed subsets $t$ of $\kappa^+$ such that
Fix a $\square^\lambda_\kappa$-sequence $\langle C_\alpha : \alpha < \kappa^+ \rangle$ and assume $\text{cof}(\kappa) > \lambda$. We shall show that $\mathcal{R}C$ fails. We consider the tree $T$ of all countable closed subsets $t$ of $\kappa^+$ such that

$$\forall \alpha \in \text{Lim}(t) \forall C \in C_\alpha \max(t \cap \text{Lim}(C)) < \alpha.$$  \hspace{1cm} (1)
Fix a □^λ_κ-sequence \( \langle C_\alpha : \alpha < \kappa^+ \rangle \) and assume \( \text{cof} (\kappa) > \lambda \). We shall show that \( \mathcal{R} \mathcal{C} \) fails. We consider the tree \( T \) of all countable closed subsets \( t \) of \( \kappa^+ \) such that

\[
\forall \alpha \in \text{Lim}(t) \forall C \in \mathcal{C}_\alpha \max(t \cap \text{Lim}(C)) < \alpha.
\] (1)

The ordering on \( T \) is by end-extension.
Claim

Every subtree of $T$ of cardinality $\aleph_1$ is special.
Proof of the Claim
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$$T(\delta) = \{ t \in T : \max(t) < \delta \}$$

contains a nonspecial subtree $U$ of size $\aleph_1$. Clearly, $\delta$ is a limit ordinal of cofinality $\omega_1$. Pick $C \in \mathcal{C}_\delta$ that is unbounded in $\delta$ and fix $D \subseteq \text{Lim}(C)$ of order-type $\omega_1$. Let
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$$D = \{ \delta_\xi : \xi < \omega_1 \}$$
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be the increasing enumeration of $D$. 

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Proof of the Claim

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be the increasing enumeration of $D$. We may assume that the nonspecial subtree $U$ of $T(\delta)$ is downward closed, so for $t \in U$ the ordinal $\xi_t = \text{otp}(t) - 1$ is its height both in $U$ and $T$. Then, by 1, for every $t \in U$ of limit height, there is $h(t) < \xi_t$ such that
Proof of the Claim

Otherwise, let $\delta < \kappa^+$ be the minimal ordinal for which the subtree

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$$(t \upharpoonright h(t)) \cap D = \emptyset.$$
Proof of the Claim

By the Pressing Down Lemma (PDL) for nonspecial trees, there is a nonspecial subtree $V$ of $U$, $\alpha_0 < \omega_1$ and $t_0 \in U$ of height $\alpha_0$ such that $h(t) = \alpha_0$ and $t \upharpoonright \alpha_0 = t_0$ for all $t \in V$. It follows that $\bigcup_{t \in V} (t \setminus t_0) = \emptyset$. 

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\[
\left( \bigcup_{t \in V} (t \setminus t_0) \right) \cap D = \emptyset.
\]
Proof of the Claim

Define now a regressive mapping $g : V \to V$ as follows. We may assume that $t_0 \in V$. Nodes smaller or equal to $t_0$ are mapped to $\emptyset$. Nodes $t \in V$ that extend $t_0$ and that have immediate predecessor in $V$ are mapped by $g$ to that predecessor. If $t \in V$ is of some limit height in $V$ and it has no predecessors $s$ with the property that $\max(s), \max(t) \notin D = \emptyset$, then we let $g(t) = t_0$; otherwise, we let $g(t)$ be the minimal predecessor $s$ of $t$ in $V$ with this property. Note that by the choice of $V$, we do have that in this case $g(t)$ is indeed a strict predecessor of $t$. Thus, $g$ is a regressive mapping on $V$.
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\[
\max(s), \max(t) \not\in D = \emptyset,
\]

then we let \( g(t) = t_0 \); otherwise, we let \( g(t) \) be the minimal predecessor \( s \) of \( t \) in \( V \) with this property. Note that by the choice of \( V \), we do have that in this case \( g(t) \) is indeed a strict predecessor of \( t \). Thus, \( g \) is a regressive mapping on \( V \).

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Proof of the Claim

Applying PDL for nonspecial trees to $g$, we get a nonspecial tree $W \subseteq V$ and $s_0 \in V$ such that $g(t) = s_0$ for all $t \in V$. Note that we must have the last alternative of the definition of $g(t)$ for $t \in W$, or in other words, we have that
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Proof of the Claim

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$$[\max(s_0), \max(t)] \cap D = \emptyset \text{ for all } t \in W.$$

Let $\alpha_0 = \max(s_0) + 1$, and let $\delta_0 = \min D \setminus \alpha_0$. Then $\delta_0 < \delta$ and $W \subseteq T(\delta_0)$ contradicting the assumption that $\delta$ is the minimal ordinal such that $T(\delta)$ contains such a nonspecial subtree of cardinality $\aleph_1$. 
It remains to prove the following:
It remains to prove the following:

Claim

*The tree $T$ is nonspecial.*
Proof of the Claim
In fact, we shall show that $T$ is a Baire tree. So let $D_n \ (n \in \omega)$ be a fixed sequence of dense-open subsets of $T$ and let $t_0 \in T$ be given. We should find $t \in T$ end extending $t_0$ and belonging to the intersection $\bigcap_{n<\omega} D_n$. 
Proof of the Claim

In fact, we shall show that $T$ is a Baire tree. So let $\mathcal{D}_n$ ($n \in \omega$) be a fixed sequence of dense-open subsets of $T$ and let $t_0 \in T$ be given. We should find $t \in T$ extending $t_0$ and belonging to the intersection $\bigcap_{n<\omega} \mathcal{D}_n$. Fix a continuous $\in$-chain $M_\xi$ ($\xi < \kappa^+$) of elementary submodels of $\langle H_{\kappa^{++}}, \in \rangle$ containing all these objects such that $M_\xi \cap \kappa^+ = \delta_\xi \in \kappa^+ \setminus \kappa$. 
Proof of the Claim

Let $E = \{ \delta \alpha : \alpha < \kappa^+, \text{cof}(\alpha) = \lambda^+ \}$. 
Let

\[ E = \{ \delta_\xi : \xi < \kappa^+, \text{cof} (\xi) = \lambda^+ \}. \]
Proof of the Claim

Choose an ordinal $\gamma < \kappa^+$ of cofinality $\omega$ such that for some increasing sequence $\gamma_n \ (n < \omega)$ converging to $\gamma$ we have that for all $n < \omega$, 

$$\text{otp}(E) \setminus (\gamma_n, \gamma_n + 1) \geq \kappa^2.$$ 

Since $\text{otp}(C) \leq \kappa$ for all $C \in C_\gamma$ and since $|C_\gamma| \leq \lambda < \text{cof}(\kappa)$, for each $n < \omega$ we can find $\delta_\xi \in E \setminus (\gamma_n, \gamma_n + 1)$ such that $(\forall C \in C_\gamma) \delta_\xi / \notin C.$
Proof of the Claim

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Choose an ordinal $\gamma < \kappa^+$ of cofinality $\omega$ such that for some increasing sequence $\gamma_n \ (n < \omega)$ converging to $\gamma$ we have that for all $n < \omega$,

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Proof of the Claim

Since $\text{cof}(\delta_{\xi_n}) = \text{cof}(\xi_n) = \lambda + \alpha_n < \delta_{\xi_n}$ (and therefore $\alpha_n \in M_{\xi_n}$) such that $\alpha_n > \delta_{\xi_n} - 1$ and $(\forall C \in C_{\gamma}) \max(C \setminus \delta_{\xi_n}) < \alpha_n$. 
Since $\text{cof } (\delta_{\xi_n}) = \text{cof } (\xi_n) = \lambda^+$ for each $n \in \omega$ we can moreover find $\alpha_n < \delta_{\xi_n}$ (and therefore $\alpha_n \in M_{\xi_n}$) such that $\alpha_n > \delta_{\xi_{n-1}}$ and
Proof of the Claim

Since \( \text{cof} (\delta_{\xi_n}) = \text{cof} (\xi_n) = \lambda^+ \) for each \( n \in \omega \) we can moreover find \( \alpha_n < \delta_{\xi_n} \) (and therefore \( \alpha_n \in M_{\xi_n} \)) such that \( \alpha_n > \delta_{\xi_{n-1}} \) and

\[
(\forall C \in \mathcal{C}_\gamma) \max (C \cap \delta_{\xi_n}) < \alpha_n.
\]
Proof of the Claim

Now, starting from $t_0$ we build an increasing sequence $t_n < \omega$ of elements of $T$ such that for all $n < \omega$, $t_n + 1 \in D_n \setminus M_{\xi_n}$. 

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$$t_{n+1} \in D_n \cap M_{\xi_n}.$$
Proof of the Claim

Given $t_n \cup \{\alpha_n\} \in M_\xi$, by the assumption that $D_n$ is open-dense in $T$ and by the elementarity of $M_\xi$, we can find $t_n^{++}$ end-extending $t_n \cup \{\alpha_n\}$ such that $t_n^{++} \in D_n \setminus M_\xi$. Let $t = \bigcup_{n < \omega} t_n \cup \{\gamma\}$.

Then $t$ is a closed subset of $\kappa$ end-extending all $t_n$'s and belonging to $t$ (and also in all $D_n$'s) since $\sup(t \setminus \text{Lim}(C)) < \gamma$ for all $C \in C_\gamma$. 

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Proof of the Claim

Given $t_n$, since $t_n \cup \{\alpha_n\} \in M_{\xi_n}$, by the assumption that $\mathcal{D}_n$ is open-dense in $T$ and by the elementarity of $M_{\xi_n}$, we can find $t_{n+1}$ end-extending $t_n \cup \{\alpha_n\}$ such that $t_{n+1} \in \mathcal{D}_n \cap M_{\xi_n}$. Let
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\[
t = \left( \bigcup_{n<\omega} t_n \right) \cup \{ \gamma \}.
\]

Then \( t \) is a closed subset of \( \kappa^+ \) end-extending all \( t_n \)'s and belonging to \( t \) (and also in all \( \mathcal{D}_n \)'s) since

\[
\sup(t \cap \text{Lim}(C)) < \gamma
\]

for all \( C \in \mathcal{C}_\gamma \).
Thanks!