Partial square and forcing axioms

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1. Square principle

**Def. (R.B. Jensen)** For an uncountable cardinal $\kappa$ let

$$\square_\kappa \equiv \text{there exists a sequence } \langle c_\alpha \mid \alpha \in \text{Lim}(\kappa^+) \rangle \text{ s.t.}

(i) \quad c_\alpha \text{ is a club subset of } \alpha \text{ with } \text{o.t.}(c_\alpha) \leq \kappa,

(ii) \quad \beta \in \text{Lim}(c_\alpha) \Rightarrow c_\beta = c_\alpha \cap \beta.

We call the above $\langle c_\alpha \mid \alpha \in \text{Lim}(\kappa^+) \rangle$ a $\square_\kappa$-sequence.

• $\square_\kappa$ is an anti-large cardinal property:

**Fact (M. Magidor)**

If every stationary $E \subseteq \{\alpha \in \kappa^+ \mid \text{cf}(\alpha) = \omega\}$ reflects to some ordinal $< \kappa^+$, then $\square_\kappa$ fails.

**Fact (S. Todorcevic)**

(1) Chang’s Conjecture $\Rightarrow \square_{\omega_1}$ fails.

(2) PFA $\Rightarrow \square_{\omega_1}$ fails.
2. Partial square principle

**Def.** For an uncountable cardinal \( \kappa \) and \( E \subseteq \text{Lim}(\kappa^+) \), let

\[
\square_\kappa(E) \equiv \text{there exists a sequence } \langle c_\alpha \mid \alpha \in E \rangle \text{ s.t.}
\]

(i) \( c_\alpha \) is a club subset of \( \alpha \) with o.t.(\( c_\alpha \)) \( \leq \kappa \),

(ii) \( \beta \in \text{Lim}(c_\alpha) \Rightarrow \beta \in E \land c_\beta = c_\alpha \cap \beta \).

We call the above \( \langle c_\alpha \mid \alpha \in E \rangle \) a \( \square_\kappa(E) \)-sequence.

- The following are easily seen:
  
  - \( \square_\kappa \iff \square_\kappa(C) \) holds for some club \( C \subseteq \kappa^+ \).
  
  - \( E \subseteq \kappa^+ \): nonstationary \( \Rightarrow \exists \bar{E} \supseteq E \text{ s.t. } \square_\kappa(\bar{E}) \) holds.
  
  - \( E \subseteq \{ \alpha \in \kappa^+ \mid \text{cf}(\alpha) = \omega \} \Rightarrow \square_\kappa(E) \) holds.
    
    (Let \( c_\alpha \) be an unbounded subset of \( \alpha \) with o.t.(\( c_\alpha \)) = \( \omega \).)
• If $\kappa$ is regular, then $\square_\kappa(E)$ holds for many stationary $E$:

**Notation** For regular cardinals $\mu < \lambda$ let

$$E_\mu^\lambda := \{ \alpha \in \lambda \mid \text{cf}(\alpha) = \mu \}.$$  

**Fact (S. Shelah)**
Suppose that $\kappa$ is a regular uncountable cardinal and that $\mu$ is a regular cardinal $< \kappa$.
Then there exists $E \subseteq \kappa^+$ such that

$$E \cap E_\mu^{\kappa^+} \text{ is stationary} \quad \& \quad \square_\kappa(E) \text{ holds.}$$
• The existence of $E$ such that
  $$E \cap E_{\kappa^+}^\kappa$$
  is stationary & $\Box_\kappa(E)$ holds

is independent:

**Theorem**

Suppose that $\kappa$ is a regular uncountable cardinal and that $\lambda$ is a weakly compact cardinal $> \kappa$.
Then there exists a forcing extension in which the following hold:

- $\kappa$ remains regular and $\lambda = \kappa^+$.
- There are no $E \subseteq \lambda$ such that
  $$E \cap E_{\kappa^+}^\lambda$$
  is stationary & $\Box_\kappa(E)$ holds.

In particular, the existence of $E \subseteq \omega_2$ such that
$$E \cap E_{\omega_2}^{\omega_2}$$
is stationary & $\Box_{\omega_1}(E)$ holds

is independent.
3. Partial square and forcing axioms

Main Theorem

**MM** \(\Rightarrow\) There exists \(E \subseteq \omega_2\) such that

\[E \cap E^\omega_2\] is stationary \& \(\square_{\omega_1}(E)\) holds.

(Theorem)

**PFA** \(\not\Rightarrow\) There exists \(E \subseteq \omega_2\) such that

\[E \cap E^\omega_2\] is stationary \& \(\square_{\omega_1}(E)\) holds.
4. Outline of Proof of Main Theorem

- We apply MM to the following poset:

**Def.** Suppose that $E \subseteq E_\omega^\omega$ and that $\vec{c} = \langle c_\alpha \mid \alpha \in E \rangle$ is a $\square_\omega(E)$-sequence. Then let $P(\vec{c})$ be the poset s.t.

- $|P(\vec{c})| = E$,
- $\alpha <_{P(\vec{c})} \beta \iff \beta \in \text{Lim}(c_\alpha)$
  \[\iff c_\alpha \text{ is an end extension of } c_\beta.\]

For a filter $G$ on $P(\vec{c})$ let

$c_G := \bigcup \{c_\alpha \mid \alpha \in G\}$. 
Suppose that \( E \subseteq E_{\omega_2}^\omega \) and that \( \vec{c} = \langle c_\alpha \mid \alpha \in E \rangle \) is a \( \square_{\omega_1}(E) \)-sequence. Suppose also that \( G \) is a filter on \( P(\vec{c}) \). Then

- \( c_G \) is club in \( \text{sup}(c_G) \),
- \( \beta \in \text{Lim}(c_G) \implies c_\beta = c_G \cap \beta. \)

**Def.** We say that \( \vec{c} \) is **unbounded** if

- \( \{ \alpha \in E \mid \text{o.t.}(c_\alpha) \geq \xi \} \) is dense in \( P(\vec{c}) \) for every \( \xi < \omega_1 \),
- \( \{ \alpha \in E \mid \text{sup}(c_\alpha) \geq \gamma \} \) is dense in \( P(\vec{c}) \) for every \( \gamma < \omega_2 \).

Suppose that \( \vec{c} \) is unbounded and that \( G \) is \( P(\vec{c}) \)-generic over \( V \). Then

- \( \text{sup}(c_G) = \omega_2^V \) \& \( \text{o.t.}(c_G) = \omega_1^V \).
Consequence of the forcing axiom for $P(\vec{c})$

Let $E$ and $\vec{c} = \langle c_\alpha \mid \alpha \in E \rangle$ be as before. Suppose that $g$ is a filter on $P(\vec{c})$ such that

$$g \cap \{ \alpha \in E \mid \text{o.t.}(c_\alpha) \geq \xi \} \neq \emptyset \quad \text{for every } \xi < \omega_1.$$  

Then

- $c_g$ is club in $\text{sup}(c_g)$, and $\text{o.t.}(c_g) = \omega_1$,
- $\beta \in \text{Lim}(c_g) \Rightarrow c_\beta = c_g \cap \beta$

Hence if we let $\gamma^* := \text{sup}(c_g)$ and $c_{\gamma^*} := c_g$, then

- $\gamma^* \in E^{\omega_2}_{\omega_1}$,
- $\langle c_\alpha \mid \alpha \in E \cup \{ \gamma^* \} \rangle$ is a $\square_{\omega_1}(E \cup \{ \gamma^* \})$-sequence.

In fact, if $\vec{c}$ is unbounded and the forcing axiom holds for $P(\vec{c})$, then there are stationary many $\gamma \in E^{\omega_2}_{\omega_1}$ such that $\vec{c}$ can be extended to $\square_{\omega_1}(E \cup \{ \gamma \})$-sequence.
Thus the following hold:

**Lemma**  Assume MM. Suppose that $\vec{c}$ is an unbounded
$\square_{\omega_1}(E^{\omega_2})$-sequence such that $P(\vec{c})$ is $\omega_1$-stationary preserving.
Then $\vec{c}$ can be extended to $\square_{\omega_1}(\bar{E})$-sequence for some $\bar{E} \subseteq \omega_2$
with $\bar{E} \cap E^{\omega_2}_{\omega_1}$ stationary.
In particular, there exists $\bar{E} \subseteq \omega_2$ such that
$$\bar{E} \cap E^{\omega_2}_{\omega_1} \text{ is stationary } \& \quad \square_{\omega_1}(\bar{E}) \text{ holds.}$$

• Therefore it suffices for Main Thm. to show the following:

$$\text{MM } \Rightarrow \text{ There exists an unbounded } \square_{\omega_1}(E^{\omega_2}) \text{-sequence } \vec{c}$$
$$\text{ such that } P(\vec{c}) \text{ is } \omega_1 \text{-stationary preserving.}$$
• We use ♦-principle:

**Fact** (M. Foreman, M. Magidor and S. Shelah)

\[ \text{MM} \implies 2^{\omega_1} = \omega_2. \]

**Fact** (S. Shelah)

\[ 2^{\omega_1} = \omega_2 \implies \lozenge_{\omega_2}(E_{\omega}^{\omega_2}). \]

Hence MM implies \( \lozenge_{\omega_2}(E_{\omega}^{\omega_2}) \). We prove the following:

**Main Lemma**

\[ \lozenge_{\omega_2}(E_{\omega}^{\omega_2}) \implies \text{There exists an unbounded } \square_{\omega_1}(E_{\omega}^{\omega_2})\text{-sequence } \vec{c} \text{ such that } P(\vec{c}) \text{ is } \omega_1\text{-stationary preserving.} \]

The proof is similar as the construction of Suslin tree using ♦.
• $\omega_1$-stationary preserving poset and projectively stationary set

**Def. (Q. Feng and T. Jech)**

Let $W$ be a set $\supseteq \omega_1$, and let $X \subseteq \mathcal{P}_{\omega_1} W$. $X$ is said to be **projectively stationary** if

for every stationary $S \subseteq \omega_1$,
the set $\{x \in X \mid x \cap \omega_1 \in S\}$ is stationary in $\mathcal{P}_{\omega_1} W$.

**Def.** Let $P$ be a poset, and $M$ be a set. $q \in P$ is said to be a **$(M, P)$-generic condition** if

for every maximal antichain $A \in M$ of $P$,
$A \cap M$ is predense below $q$.

**Lemma**  Assume that $P$ is a poset with the following property:

For every $p \in P$ and every sufficiently large reg. cardinal $\theta$,

$\{M \in \mathcal{P}_{\omega_1} H_{\theta} \mid (M, P)$-generic condition below $p$ exists\}$
is projectively stationary.

Then $P$ is $\omega_1$-stationary preserving.
• ♦_{\omega_2}(E^\omega_2) implies the following diamond principle in \mathcal{P}_{\omega_1\omega_2}:

**Lemma** If ♦_{\omega_2}(E^\omega_2) holds, then the following holds:

There exist a stationary \(X \subseteq \mathcal{P}_{\omega_1\omega_2}\) and a sequence 
\(\langle \alpha_x, B_x \mid x \in X \rangle\) with the following properties:

(i) \(X\) is skinniest

i.e. \(|\{x \in X \mid \sup(x) = \alpha\}| \leq 1\) for every \(\alpha \in \omega_2\).

(ii) \(\alpha_x \in x \cap E^\omega_2\).

(iii) \(B_x\) is a countable subset of \(\mathcal{P}(x)\).

(iv) For every suff. large reg. \(\theta\) and every \(\alpha \in E^\omega_2\),

the set of all \(M \in \mathcal{P}_{\omega_1\mathcal{H}_\theta}\) such that

- \(x^* := M \cap \omega_2 \in X\),
- \(\alpha = \alpha_{x^*}\),
- \(\{B \cap x^* \mid B \in \mathcal{P}(\omega_2) \cap M\} = B_{x^*}\),

is projectively stationary in \(\mathcal{P}_{\omega_1\mathcal{H}_\theta}\).
Outline of Proof of Main Lemma

Assume $\Diamond_{\omega_2}(E^{\omega_2}_{\omega_2})$ holds.
Let $X$ and $\langle \alpha_x, B_x \mid x \in X \rangle$ be as in the previous lemma.

By induction on $\alpha \in E^{\omega_2}_{\omega_2}$ we construct a $\square_{\omega_1}(E^{\omega_2}_{\omega_2})$-sequence $\langle c_\alpha \mid \alpha \in E^{\omega_2}_{\omega_2} \rangle$.
Suppose that $\alpha \in E^{\omega_2}_{\omega_2}$ and that $\langle c_\beta \mid \beta \in E^{\omega_2}_{\omega_2} \cap \alpha \rangle$ has been defined.

Case 1  There are no $x \in X$ with $\sup(x) = \alpha$.

Let $c_\alpha$ be an arbitrary unbounded subset of $\alpha$ of order type $\omega$. 
Case 2  There exists $x \in X$ with $\sup(x) = \alpha$.

(Such $x$ is unique because $X$ is skinniest.)

Consider the poset $P_\alpha := P(\langle c_\beta | \beta \in x \cap E_{\omega^2} \rangle)$.

First take a filter $g_\alpha$ on $P_\alpha$ such that
- $\alpha_x \in g_\alpha$,
- $g_\alpha \cap b \neq \emptyset$ for every $b \in B_x$ which is a max. antichian in $P_\alpha$.

(Such $g_\alpha$ can be taken because $B_x$ is countable.)

If $c_{g_\alpha}$ is unbounded in $\alpha$, then let $c_\alpha := c_{g_\alpha}$.

Otherwise take an unbounded $c \subseteq \alpha$ such that $\min(c) > \alpha_x$ and $\text{o.t.}(c_\alpha) = \omega$, and let $c_\alpha := c_{\alpha_x} \cup c$.

Note that if $M \cap \omega_2 = x$, $B_x = \{ B \cap x \mid B \in \mathcal{P}(\omega_2) \cap M \}$ and $c_\alpha = c_{g_\alpha}$, then $\alpha$ will be a $M$-generic condition.

Note also that $\alpha$ will be an extension of $\alpha_x$ in the poset.
Let \( \vec{c} := \langle c_\alpha \mid \alpha \in E^{\omega_2}_\omega \rangle \).

Now it is not hard to see the following:

- \( \vec{c} \) is \( \square_{\omega_1}(E^{\omega_2}_\omega) \)-sequence.

- For every \( \alpha \in E^{\omega_2}_\omega \),
  \( \{ M \in P_{\omega_1}H_\theta \mid (M, P(\vec{c})) \text{-generic condition below } \alpha \text{ exists} \} \)
  is projectively stationary.

By the second property \( P(\vec{c}) \) is \( \omega_1 \)-stationary preserving.

It is not hard to see that \( \vec{c} \) is unbounded, too. \( \square \)
**Question**  How about partial square at singular $\kappa$?  
For example, 

$$\text{MM} \implies \text{There exists } E \subseteq \omega_{\omega+1} \text{ such that}$$

$$E \cap E_{\omega_1}^{\omega+1} \text{ is stationary } \& \Box_{\omega}(E) \text{ holds.}$$