Borel Complexity of Isomorphism Between Quotient Boolean Algebras

Su Gao

Department of Mathematics
University of North Texas
sgao@unt.edu
http://www.math.unt.edu/sgao/

29 August 2008
RIMS
Joint work with Mike Oliver

To appear in the Journal of Symbolic Logic

Preprint can be found at http://www.math.unt.edu/~sgao/pub/
Question (Farah)
How many Boolean algebras $\mathcal{P}(\omega)/I$ are there?

Answer
$2^{2^{\aleph_0}}$, since every complete Boolean algebra of size $2^{\aleph_0}$ is isomorphic to one of the form $\mathcal{P}(\omega)/I$, and Monk-Solovay counted that there are $2^{2^{\aleph_0}}$ many such Boolean algebras.
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Oliver (2004) constructed $2^{\aleph_0}$ many Borel ideals with pairwise nonisomorphic quotients.
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An ideal $I$ on $\omega$ is Borel if it is a Borel subset of $\mathcal{P}(\omega) = 2^\omega$ (the Cantor space with the product topology).
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1. \( \text{Fin} = \) the ideal of all finite subsets of \( \omega \)
2. \( \mathcal{Z}_0 = \) the density ideal

\[ X \in \mathcal{Z}_0 \iff \lim_{n \to \infty} \frac{|X \cap [0, n]|}{n + 1} = 0 \]
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$\text{Fin}$ is $F_\sigma$ or $\Sigma^0_2$, $\mathcal{Z}_0$ is $\Pi^0_3$. 
Examples

1. $\text{Fin} \times \emptyset =$ the ideal of all subsets of $\omega \times \omega$ with only finitely many nonempty vertical sections

2. $\emptyset \times \text{Fin} =$ the ideal of all subsets of $\omega \times \omega$ with no infinite vertical sections

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**Fact** (Just-Krawczyk)

Assume CH. If $I$ is $\Sigma^0_2$ then $\mathcal{P}(\omega)/I \cong \mathcal{P}(\omega)/\text{Fin}$. 
Definition

If $I, J$ are ideals on $\omega$, define $I \times J$ on $\omega \times \omega$ by

$$A \in I \times J \iff \{n \mid \{m \mid (n, m) \in A\} \notin J\} \in I.$$
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Definition
If $I, J$ are ideals on $\omega$, then they are Rudin-Kiesler isomorphic, denoted $I \cong_{RK} J$, if there are $A, B \subseteq \omega$, with $\omega \setminus A \in I$ and $\omega \setminus B \in J$, and a bijection $f : A \to B$, such that for all $C \subseteq A$,

$$C \in I \iff f(C) \in J.$$
Theorem (Oliver)

*There exist Π\(^0_3\) ideals \(\overline{J}_x, x \in 2^\omega\), on \(\omega\) such that*

1. if \(x \neq y\), \(X, Y \subseteq \omega\) with \(X \notin \overline{J}_x\) and \(Y \notin \overline{J}_y\), then \(\overline{J}_x \upharpoonright X \not\equiv_{RK} \overline{J}_y \upharpoonright Y\);

2. letting \(J_x = (\overline{J}_x \times \emptyset) \cap (\emptyset \times \text{Fin})\), then for \(x \neq y\),

\[
\mathcal{P}(\omega \times \omega)/J_x \not\equiv \mathcal{P}(\omega \times \omega)/J_y.
\]
Equivalence Relations

Mission Impossible
How to obtain “more than” continuum many Borel ideals with pairwise nonisomorphic quotients?
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How to obtain “more than” continuum many Borel ideals with pairwise nonisomorphic quotients?

Answer
Use equivalence relations that are known to be more complex than the equality of real numbers.
Definition
Let $E, F$ be equivalence relations on spaces $X, Y$, respectively. We say that $E$ is reducible to $F$, denoted $E \leq F$, if there is a function $f : X \to Y$ such that

$$x_1 E x_2 \iff f(x_1) F f(x_2).$$
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When $X, Y$ are standard Borel spaces, and $f : X \to Y$ a Borel function, we say that $E$ is **Borel reducible** to $F$, and denote by $E \leq_B F$. 

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$$E \leq_B F = E \text{ is no more complex than } F$$
Examples

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\[ x E_v y \iff x - y \in \mathbb{Q}. \]
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\[ \text{id}(2^\omega) \leq_B \text{id}(\mathbb{R}) \leq_B \text{id}(2^\omega) \leq_B E_v \nleq_B \text{id}(2^\omega) \]

\[ \text{id}(2^\omega) \sim_B \text{id}(\mathbb{R}) <_B E_v \]
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**Theorem (Gao-Oliver)**

*If $F$ is a Borel equivalence relation, then $F \leq_B E$.***
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**Theorem (Gao-Oliver)**

*If $F$ is a Borel equivalence relation, then $F \leq_B E$.***

**Theorem**

*There exist Borel ideals $J_A$, where $A$ ranges over all Borel subsets of $2^\omega$, such that for Borel $A, B \subseteq 2^\omega$,

$$\mathcal{P}(\omega)/J_A \cong \mathcal{P}(\omega)/J_B \iff A = B.$$*
Amalgam of Borel Ideals

The Adams-Kechris Technique

Theorem (Adams-Kechris)

There exist countable Borel equivalence relations $E_A$, where $A$ ranges over all Borel subsets of $2^\omega$, such that for Borel $A, B \subseteq 2^\omega$,

$$E_A \leq_B E_B \iff A \subseteq B.$$  

It follows that

$$E_A \sim_B E_B \iff A = B.$$
Theorem (Clemens)

There exist Borel automorphisms $\varphi_A$ on $2^\omega$, where $A$ ranges over all Borel subsets of $2^\omega$, such that for Borel $A, B \subseteq 2^\omega$,

$$
\varphi_A \approx \varphi_B \iff A = B.
$$
Theorem (Gao)

*If* $F$ *is any analytic equivalence relation, then* $F$ *is Borel reducible to*

- the Borel bireducibility relation between countable Borel equivalence relations, or
- the conjugacy equivalence between Borel automorphisms.
Adams-Kechris made use of a family of “strongly mutually ergodic” family of equivalence relations $E_x, x \in 2^\omega$.

They then defined

$$E_A = \bigsqcup_{x \in A} E_x.$$
Two Challenges

1. In amalgamating continuum many ideals into a single ideal on $\omega$, it is impossible to accommodate all of them without allowing interactions.

2. Assume CH. For any ideals $I, J$ on $\omega$, $\mathcal{P}(\omega)/I \hookrightarrow \mathcal{P}(\omega)/J$. 
An idea due to Hjorth

Given a Borel $A \subseteq 2^\omega$, we will “glue together” ideals of the form $J_x$ (recall that these are ideals on $\omega \times \omega$) for all $x \in A$ to produce an ideal $I_A$. 

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Formally,

$$X \in I_A \iff (\forall x \in A)\{(n, m) \mid \exists^\infty k \ (n, m, x | k) \in X\} \in J_x.$$
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Given a Borel $A \subseteq 2^\omega$, we will “glue together” ideals of the form $J_x$ (recall that these are ideals on $\omega \times \omega$) for all $x \in A$ to produce an ideal $I_A$.

$I_A$ will live on $\omega \times \omega \times 2^{<\omega}$. Think of $2^{<\omega}$ as approximating $2^\omega$. For $X \subseteq \omega \times \omega \times 2^{<\omega}$, close the $2^{<\omega}$ under subsequence (to form a tree). Look at levels that have $x$ as a branch. Is the set of those levels in $J_x$?

Formally,

$$X \in I_A \iff (\forall x \in A)\{(n, m) \mid \exists k (n, m, x \upharpoonright k) \in X\} \in J_x.$$ 

The problem with $I_A$ is that they are apparently coanalytic and perhaps non-Borel. To address this issue we need one more idea.
Define the finite-threaded ideal on $\omega \times \omega \times 2^{<\omega}$ by

$$X \in \text{FT} \iff (\exists x_0, \ldots, x_t \in 2^\omega) \ X \subseteq \bigcup_{0 \leq i \leq t} 1_{x_i},$$

where

$$1_x = \{(n, m, x|k) \mid n, m, k \in \omega\}.$$
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FT is $\Sigma^0_2$ since $X \in FT$ iff

$$(\exists t)(\forall (n_0, m_0, s_0), \ldots, (n_t, m_t, s_t) \in X)(\exists i \neq j \leq t)$$

$$s_i \subseteq s_j \text{ or } s_j \subseteq s_i$$
We finally let

\[ J_A = I_A \cap FT. \]

Then \( J_A \) is Borel.
Lemma
Let $X \subseteq 1_x$ and $Y \subseteq 1_y$. Suppose $X \notin \text{SASI}(J_A)$ and
\[ \mathcal{P}(X)/J_A \cong \mathcal{P}(Y)/J_B. \]
Then there are $X', Y' \subseteq \omega$ with $X' \notin \overline{J}_x$, $Y' \notin \overline{J}_y$, such that
\[ \overline{J}_x \upharpoonright X' \cong_{RK} \overline{J}_y \upharpoonright Y'. \]
In particular $x = y$. 

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**ω-partition:** a sequence $A_0, A_1, \cdots \not\in I$ of disjoint sets so that $A$ is the sup of $A_i$ wrt $I$, that is, $A_i \setminus A \in I$ for all $i \in \omega$, and if $A_i \setminus B \in I$ for all $i$ then $A \setminus B \in I$. 
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$I$ is shallow: every non-$I$ set has an ω-partition wrt $I$ (Fin is not shallow)
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$I$ is shallow: every non-$I$ set has an $\omega$-partition wrt $I$ (Fin is not shallow)

\[ \text{SI}(I) = \text{the ideal of all } A \text{ such that } I \upharpoonright A \text{ is shallow} \]

\[ \text{ASI}(I) = \text{the ideal of all } A \text{ such that } A \in \text{SI}(I) \iff A \in I \]

\[ \text{SASI}(I) = \text{the ideal of all } A = B \cup C, \text{ where } B \in \text{SI}(I) \text{ and } C \in \text{ASI}(I) \]
Suppose $x \in A \setminus B$ but

$$\phi : \mathcal{P}(\omega \times \omega \times 2^{<\omega})/J_A \cong \mathcal{P}(\omega \times \omega \times 2^{<\omega})/J_B.$$
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If \( \phi(1_x) \subseteq \bigcup_{j<i} 1_{y_j} \) then \( \phi(1_x) \notin \text{FT} \).
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If $\phi(1_x) \subseteq \bigcup_{j<i} 1_{y_j}$ then $\phi(1_x) \not\in \text{FT}$. 

Otherwise we may construct $Z \subseteq \mathcal{P}(\omega \times \omega \times 2^{<\omega})$ such that

- $Z \subseteq \phi(1_x)$,
- the projection of $Z$ to the “tree coordinate” is an antichain in $2^{<\omega}$, and
- each $1_y$ contains at most one element of $Z$. 

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These imply that $Z \in \text{Sl}(J_B)$ on the one hand and $J_B \upharpoonright Z \cong \text{Fin}$ on the other, but Fin is not shallow!
Further Questions

Can one construct $\Pi^0_3$ ideals to fulfill this?

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- Is every analytic equivalence relation reducible to the isomorphism of quotient Boolean algebras by Borel ideals?
Thank you, Jörg!

for all the hard work you have done

for a great workshop