Large Cardinals
and
L-Like Combinatorics

or, “No Worries!”

Andrew Brooke-Taylor

July 23, 2007

Acknowledgements

It has been a long road to the production of this dissertation, and I have many people to thank.

I have been privileged to have not one but three great advisors. Haynes Miller was very patient with me while I was finding my way to my true mathematical calling. Gerald Sacks took me in and helped me start my quest to become a set theorist, with sage advice and his inimitable sense of humour. Of course, I would especially like to thank Sy Friedman, for bringing me to Vienna and guiding me through the writing of the thesis, being at once both a strong motivating force and friendly mentor.

I would also like to acknowledge mathematical guidance I have received from other people as well. In Boston, Akihiro Kanamori and Peter Koellner were constantly answering questions I had, and the other logic students in Boston — Nate, Alice, Christina, Cameron, Josh and David — formed a wonderful community in which to study logic. In Vienna, these roles have been filled by everyone at the Kurt Gödel Research Center, along with Martin Goldstern at TU Wien. I should particularly thank Heike Mildenberger for always being happy to have me barge in and ask questions, and Jakob Kellner for encouraging me to pursue variants of $\diamondsuit$ as possible coding oracles, alongside all the many other reasons I have for thanking both of them.

I would like to thank Mum, Dad, Michael and Sallyanne for all their support, along with my extended family who have been there for me through the years: my close English and Australian relatives, the Kirby-Joneses, my surrogate parents in Boston Ken and Bonny, and also Martina's family who have treated me as one of their own.

The past six years have also been a lot of fun. In addition to those already mentioned above, I would particularly like to thank Shelby, Brian, Michael, Heather, Karen, Dirk, Damiano, Ilya, Bianca, Anna, Reimundo, Mariana, Miki, Ljudmilla, Sandra, John, Katie, Lydia, Katie, Chris, Agatha, Rainer, Natasha, John, Matteo, David, Radek, Tomáš, Pavel, Richard, Max, Louise, Roman, William, Regina, Ayda, Johanna, Kathryn, Amy and Jo for all the good times. Finally, thank you to Martina for making this last year such a

wonderful one.

Thank you all.

Contents

Introduction iii
0.1 A note on terminology iv
0.2 A note on the subtitle v

1 Preliminaries 1
1.1 General notation 1
1.2 Large cardinals 2
1.3 Class forcing 5
1.4 Lifting Embeddings 13

2 GCH 15
2.1 Forcing the GCH 15
2.2 1-extendible cardinals and the GCH 17

3 Morasses and Mangroves 23
3.1 What they are 23
3.2 Forcing them to exist 26
  3.2.1 Definitions 26
  3.2.2 A first “bamboo construction” 28
  3.2.3 μ-equivalence 29
  3.2.4 Cardinal preservation 32
  3.2.5 A mangrove from a generic 38
3.3 A mangrove from a morass 43
  3.3.1 Aside: Complete embeddings 47
3.4 Homogeneity 50
3.5 Preserving large cardinals 55
3.6 1-extendible cardinals and morasses 58
3.7 Universal morasses 60

4 Definable Well-Ordering 71
4.1 Forcing $\diamondsuit^*_{\kappa^+}$ to hold or fail 71

4.1.1 Forcing $\diamondsuit^*_{\kappa^+}$ to hold 72
4.1.2 Forcing $\diamondsuit^*_{\kappa^+}$ to fail 79
4.2 Forcing a definable well-order 82
4.3 Preserving large cardinals 85

Introduction

Gödel's constructible universe L satisfies a myriad of useful and aesthetically pleasing combinatorial principles, including the GCH, $\diamondsuit$, the existence of morasses, and the existence of a definable well-order of the whole of L. On the other hand, L cannot tolerate the existence of many of the large cardinals that one would like to consider. The question arises as to how L-like the universe can be while still containing such large cardinals. One approach to this problem is given by the *inner model programme*, in which canonical inner models containing the large cardinals are constructed within any universe which itself has large cardinals of that kind. This programme has had much success, but there is a limit to the strength of large cardinal axioms that can currently be accommodated by such models.

An alternative is given by the *outer model programme*, whereby we again start with a model containing very large cardinals, but now use iterated forcing to obtain an L-like *outer* model. In [8] Friedman shows that each of the above-mentioned principles may be obtained by forcing while preserving a superstrong, hyperstrong, n-superstrong or ω-superstrong cardinal. Work has also been done regarding the principle $\square$, which is an interesting case, because it is compatible with some very large cardinals but not others, with 1-*extendible cardinals* seeming to form an upper bound — see [3], [4] and [8].

The aim of this dissertation is to extend the work of [8], focussing on the existence of (gap 1) morasses at every uncountable regular cardinal, and the existence of a definable well-order of the universe. In particular, we show that each may be forced while simultaneously preserving a proper class of superstrong, hyperstrong, n-superstrong and ω-superstrong cardinals. In the morasses case, this entails modifying the usual Velleman forcing (as in [16]) so that it enjoys more homogeneity, and in the process, we are naturally directed to consider a particular class of morasses, which we name *mangroves*. In the definable well-order of the universe case, an entirely new forcing is needed. In particular, our forcing overcomes the problems associated with having many different master conditions by simply avoiding the need for master conditions altogether. This makes it straightforward to also preserve other "local" large cardinals, witnessed by only set-many embeddings, such as η-extendible or $\kappa^{+\eta}$-supercompact cardinals.

In the case of morasses, we also show with some extra effort that 1-extendible cardinals may be preserved — a careful analysis is needed in this case because the forcing partial order in question is class-sized relative to the domain of the embedding of interest. We show that if the class forcing to obtain the GCH is applied first, these technical problems will be resolved. Also in the morasses chapter, we consider universal morasses, which carry with them subsets of κ^+ coding up every subset of κ. We show how to force such objects to exist, and that this forcing may be carried out while preserving all superstrong cardinals and a given hyperstrong or m-superstrong cardinal, for any $m \in \omega + 1$. The question of whether universal morasses can exist in the presence of large cardinals was brought to our attention by Pereira, with applications to the construction of scales in pcf theory in mind.

0.1 A note on terminology

A few words are perhaps in order regarding the terms “mangal” and “mangrove” introduced in Chapter 3. An obvious first choice of name for a variant of a morass was *quagmire*, but this has already been used by Burgess ([2]; see also [12]) for a weakening of the notion of morass. So instead, I moved closer to home, naming the structures after the only kind of bog I can remember walking through — a mangrove, a kind of tropical, coastal swamp.

Whilst the word “mangrove” does not have the same common-usage meaning of “intractable mess” that “morass” and “quagmire” enjoy, it does have the benefit of being used variously to mean either the whole boggy swamp, or a single tree of the kind that lives in such an area. After all, a (mathematical) morass is a tree with (admittedly significant) additional structure.

A mangrove will generally have various zones with differing flora and fauna, dependent on the frequency with which the zone is inundated with salt water in a high tide. Thus the term “mangal”, which designates the ecological system of the mangrove, seemed a reasonable name for the important levels of our mangroves. It should be noted that for real world mangrove swamps, the term is generally used for the whole ecological system, spanning all the zones, and so perhaps “high water mark” would have fit the analogy slightly better. However, this phrase was clearly too cumbersome to be a serious contender. Also, “mangal” has the thoroughly appropriate homophone “mangle” — an object whose reason for being is to have things pass through it — and so “mangal” won the day. So now, if your morass has mangals all through it, it's a mangrove.

0.2 A note on the subtitle

The subtitle "No Worries" stems primarily (although not entirely) from the experience of working on the definable well-order part of this dissertation. At first the coding of a well-order of the universe into a subclass of the ordinals was envisaged as a complex iteration with careful bookkeeping, so the solution of "let the generic make the choices" seemed thoroughly lackadaisical by comparison. Then the idea for how to make sure all n-superstrong cardinals are preserved — just avoid those regions completely and code elsewhere — gave the feeling that we were able to just sidestep all of the perceived problems. The raising of the issue that new superstrongs could be introduced making the class of coding points non-absolute, and its resolution in the form "it doesn't matter", added to this impression. As a result of all of this, "No Worries" seemed an appropriate alternate title for this work.

Chapter 1

Preliminaries

For the sake of self-containment, we gather in this chapter various standard definitions and results that will be assumed throughout this dissertation.

1.1 General notation

We shall denote by ZFC^- the axioms of ZFC with the Power Set Axiom omitted. The class of ordinals shall be denoted Ord.

We will use the term *partial order* in the liberal sense as in [14], not requiring them to satisfy antisymmetry. When forcing, $p \leq q$ will mean that p is a stronger condition than q, with $q \Vdash \varphi$ implying that $p \Vdash \varphi$.

To denote the set of functions from A to B, we use the notation ${}^{A}B$, as in [6] and [13] to cite two prominent sources. This will serve to distinguish from ordinal exponentiation, used heavily in Chapter 3, and cardinal exponentiation, used throughout. For an ordinal α and set B, ${}^{<\alpha}B$ will denote $\bigcup_{\beta<\alpha} {}^{\beta}B$, mirroring the definition for cardinals κ and λ that $\kappa^{<\lambda} = |\bigcup_{\text{cardinals } \mu<\lambda} \kappa^{\mu}|$. Of course, there is no need for such notation for ordinal exponentiation, as by definition, $\alpha^{\gamma} = \sup_{\beta<\gamma}(\alpha^{\beta})$ for ordinals α and γ with γ a limit ordinal. For a cardinal κ and ordinal δ we shall sometimes use the notation $\kappa^{+\delta}$ to denote the δ-fold cardinal successor of κ, that is, if $\kappa = \aleph_{\alpha}$, $\kappa^{+\delta} = \aleph_{\alpha+\delta}$.

It will be convenient to use the following notation from [14].

Definition 1. *For any cardinal κ and sets I and J with $|I| \geq \kappa$, the partial order $Fn(I, J, \kappa)$ has as conditions partial functions from I to J with domain of cardinality less than κ, with the partial order relation $p \leq q \leftrightarrow p \supseteq q$.*

The following is standard — see for example Lemma VII 6.10 of [14].

Lemma 2. *For any cardinal κ and sets I and J,* $\mathrm{Fn}(I, J, \kappa)$ *has the* $(|J|^{<\kappa})^{+}$*-chain condition.*

1.2 Large cardinals

Definition 3. *For any $n \in \omega + 1$, an embedding j from V to an inner model M is* n-superstrong *if $V_{j^n(\kappa)} \subset M$, where κ is the critical point of j and $j^\omega(\kappa)$ denotes* $\sup_{0<n<\omega}(j^n(\kappa))$. *A cardinal κ is* n-superstrong *if it is the critical point of an n-superstrong embedding. A cardinal or embedding is simply said to be* superstrong *if it is 1-superstrong.*

Note that the statement "There is an ω-superstrong κ" is sometimes referred to as *Axiom I2*, for example in [13].

The notion of n-superstrength can be formalised in ZFC in terms of *extenders*; see for example [13], Chapter 26 for details beyond the overview we give here. An extender is a particular kind of system of ultrafilters such that the corresponding ultrapowers forms directed system. The direct limit of this system is referred to as the *extender ultrapower*, and if our extender is chosen appropriately, the embedding from V to the extender ultrapower will also be n-superstrong. We can nicely characterise the elements of the extender ultrapower according to the following lemma, which will be useful in our construction of various generics.

Lemma 4. *If κ is n-superstrong, there is an inner model M and an n-superstrong embedding $j : V \longrightarrow M$ with critical point κ such that every element of M is of the form $j(f)(a)$ for some $a \in V_{j^n(\kappa)}$ and $f \in V$ a function with domain $V_{j^n(\kappa)}$.*

Definition 5. *A cardinal κ is* hyperstrong *if there is an inner model M and a an elementary embedding $j : V \longrightarrow M$ with critical point κ such that $V_{j(\kappa)+1} \subset M$.*

Lemma 6. *If κ is hyperstrong, there is an inner model M and an hyperstrong embedding $j : V \longrightarrow M$ with critical point κ such that every element of M is of the form $j(f)(a)$ for some $a \in V_{j(\kappa)+1}$ and $f \in V$ a function with domain $V_{j(\kappa)}$.*

Recall that for any cardinal κ, H_κ denotes the set of all sets x such that $|\mathrm{trcl}(x)| < \kappa$.

Definition 7. *A cardinal κ is said to be* 1-extendible *if there is a $\lambda > \kappa$ and an elementary embedding $j : H_{\kappa^+} \longrightarrow H_{\lambda^+}$ with critical point κ.*

Note in particular that because $H_{\mu^+} \vDash$ "μ is the greatest cardinal", $j(\kappa) = \lambda$.

Some sources define 1-extendibility by the equivalent statement of the following well-known result.

Proposition 8. *A cardinal κ is 1-extendible if and only if there is a $\lambda > \kappa$ and an elementary embedding $k : V_{\kappa+1} \to V_{\lambda+1}$ with critical point κ.*

Proof. For the forward direction, observe first that $V_{\kappa+1}$ is a definable class in H_{κ^+}, given by

$$V_{\kappa+1} = \{x \in H_{\kappa^+} \mid x \subseteq V_\kappa\}.$$

Of course we are using here the fact that $V_\kappa \in H_{\kappa^+}$, since the standard arguments give that κ is measurable, and hence inaccessible. Also, $V_{\lambda+1}$ is definable in H_{λ^+} in the same way, with κ replaced by λ. Hence, any formula may be relativised to $V_{\kappa+1}$ within H_{κ^+}, and we see that $j|V_{\kappa+1}$ is elementary from $V_{\kappa+1}$ to $V_{\lambda+1}$.

In the other direction, suppose $k : V_{\kappa+1} \to V_{\lambda+1}$ is elementary. Note that not only is κ in $V_{\kappa+1}$, but also $\kappa \times \kappa$ and any subset of it. Now, for any element x of H_{κ^+} and any bijection $f : |\mathrm{trcl}(\{x\})| \to \mathrm{trcl}(\{x\})$, let

$$C_{x,f} = \{\langle \alpha, \beta \rangle \in \kappa \times \kappa \mid f(\alpha) \in f(\beta)\}.$$

Then $C_{x,f}$ is in $V_{\kappa+1}$, and the Mostowski collapse of $C_{x,f}$ is x. Let $\bar{H}$ denote the class in $V_{\kappa+1}$ of all subsets of $\kappa \times \kappa$ which code an element of H_{κ^+} in this way; that is, $X \in \bar{H}$ if and only if X is a well founded, extensional binary relation on κ which has a single maximal element, and the union of the domain and range of X is a cardinal less than or equal to κ. In particular, $\bar{H}$ is definable in $V_{\kappa+1}$.

For Z in $\bar{H}$, let us denote by $\mathrm{fld}(Z)$ the *field* of Z, that is, the union of the domain and range of Z; and denote by $\max(Z)$ the unique element of the field of Z which is maximal with respect to the relation Z. Now, for X and Y in $\bar{H}$, say that

$$\begin{aligned} X \equiv Y \quad \longleftrightarrow \quad & \exists f : \kappa \to \kappa \big(f \text{ is a bijection } \wedge \\ & \qquad \forall \alpha, \beta < \kappa (\langle \alpha, \beta \rangle \in X \leftrightarrow \langle f(\alpha), f(\beta) \rangle \in Y) \big), \end{aligned}$$

and

$$\begin{aligned} X \bar{\in} Y \quad \longleftrightarrow \quad & \exists f : \kappa \to \kappa \big(f \text{ is one-to-one } \wedge \\ & \qquad \forall \alpha, \beta \in \mathrm{fld}(X) (\langle \alpha, \beta \rangle \in X \leftrightarrow \langle f(\alpha), f(\beta) \rangle \in Y) \\ & \qquad \wedge \ \langle f(\max(X)), \max(Y) \rangle \in Y \big). \end{aligned}$$

Then $\equiv$ and $\bar{\in}$ are definable in $V_{\kappa+1}$, and letting $\mathrm{coll}(Z)$ denote the Mostowski collapse of Z, $X \equiv Y \leftrightarrow \mathrm{coll}(X) = \mathrm{coll}(Y)$ and $X \bar{\in} Y \leftrightarrow \mathrm{coll}(X) \in \mathrm{coll}(Y)$. Thus, any first order statement φ about H_{κ^+} is equivalent to a formula $\bar{\varphi}$ over $V_{\kappa+1}$, where parameters x are replaced by codes for them $C_{x,f}$, $=$ and $\in$ are replaced by $\equiv$ and $\bar{\in}$ respectively, and quantification is taken over $\bar{H}$.

Of course, all of the same definitions may be made with κ replaced by λ, and so by the elementarity of k, we have that the $\bar{H}$ of $V_{\kappa+1}$ is taken by k to the $\bar{H}$ of $V_{\lambda+1}$, and $X \equiv Y \leftrightarrow k(X) \equiv k(Y)$ and $X \bar{\in} Y \leftrightarrow k(X) \bar{\in} k(Y)$. Thus, we may define a mapping $j : H_{\kappa^+} \to H_{\lambda^+}$ by $j(x) = \mathrm{coll}(k(C_{x,f}))$ for some (any) bijection $f : |\mathrm{trcl}(\{x\})| \to \mathrm{trcl}(\{x\})$, and it will be elementary, since

$$\begin{aligned} H_{\kappa^+} \vDash \varphi(x_1, ..., x_n) &\longleftrightarrow V_{\kappa+1} \vDash \bar{\varphi}(C_{x_1,f_1}, ..., C_{x_n,f_n}) \\ &\longleftrightarrow V_{\lambda+1} \vDash \bar{\varphi}(k(C_{x_1,f_1}), ..., k(C_{x_n,f_n})) \\ &\longleftrightarrow H_{\lambda^+} \vDash \varphi(\mathrm{coll}(k(C_{x_1,f_1})), ..., \mathrm{coll}(k(C_{x_n,f_n}))). \end{aligned}$$

Hence, κ is 1-extendible, as defined above. ⊣

It will be useful for us that H_{κ^+} is a model of ZFC$^-$,and for this reason we will generally work with 1-extendibility using the $H_{\kappa^+} \to H_{\lambda^+}$ formulation.

Using satisfaction predicates, it is clear that any j witnessing the hyperstrength of κ will also witness the ($V_{\kappa+1} \to V_{\lambda+1}$ form of) the 1-extendibility of κ, when restricted to $V_{\kappa+1}$. On the other hand, it turns out that any 1-extendible κ is superstrong, and moreover has many superstrongs below it; see [13], §26.

We now list some further large cardinals that will be mentioned at the end of Chapter 4, along with the corresponding representation lemmas. In each case, details may be found in [13].

Definition 9. *A cardinal κ is* measurable *if there is an inner model M and an elementary embedding $j : V \to M$ with critical point κ.*

Lemma 10. *If κ is measurable, there is an inner model M and a elementary embedding $j : V \to M$ with critical point κ such that every element of M is of the form $j(f)(\kappa)$ for some $f : \kappa \to V$ in V.*

Definition 11. *A cardinal κ is η-*strong *for an ordinal η if there is an inner model M and an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \eta$ and $V_{\kappa+\eta} \subset M$.*

Lemma 12. *If κ is η-strong, there is an inner model M and an η-strong embedding $j : V \to M$ with critical point κ such that every element of M is of the form $j(f)(a)$ for some $a \in [|V_{\kappa+\eta}|^+]^{<\omega}$ and $f \in V$ a function with domain $[V_\kappa]^{|a|}$.*

Definition 13. *A cardinal κ is η-*extendible *for an ordinal η if there is some ordinal ζ and an elementary embedding $j : V_{\kappa+\eta} \to V_\zeta$ with critical point κ such that $j(\kappa) > \eta$.*

Definition 14. *A cardinal κ is* λ-supercompact *for an ordinal λ if there is an inner model M and an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and ${}^{\lambda}M \subseteq M$.*

Lemma 15. *If κ is λ-supercompact, there is an inner model M and an λ-supercompact embedding $j : V \to M$ with critical point κ such that every element of M is of the form $j(f)(j``\lambda)$ for some $f \in V$ a function with domain $\mathcal{P}_\kappa\lambda$.*

Definition 16. *A cardinal κ is* m-huge *for $m \in \omega$ if there is an inner model M and an elementary embedding $j : V \to M$ with critical point κ such that ${}^{j^m(\kappa)}M \subseteq M$.*

Lemma 17. *If κ is m-huge, there is an inner model M and an m-huge embedding $j : V \to M$ with critical point κ such that every element of M is of the form $j(f)(j``(j^m(\kappa)))$ for some $f \in V$ a function with domain $\mathcal{P}(j^m(\kappa))$.*

1.3 Class forcing

Our forcing constructions will be class-sized iterations, and so a measure of extra care is needed. We discuss the details here, taking definitions and results from [9], and reordering and recasting them to better fit our context. In particular, we attempt to simplify matters by omitting where possible the class predicates accompanying ZFC models which are allowed in that reference, and assuming that our models all satisfy the Axiom of Choice.

The first point of concern is the definability of the $\Vdash$ relation. For a set forcing P we define the relation $\Vdash^*$ on $P \times V^P \times \{``\in", ``="\} \times V^P$, by recursion on the appropriate well-founded relation on the last three components, and then show that $\Vdash = \Vdash^*$. Of course, the recursion defining $\Vdash^*$ is translated into a formal definition by means of "initial approximations" to the relation $\Vdash^*$: let e stand for either "$\in$" or "$=$", then

$$\begin{aligned} &p \Vdash^* \sigma e \tau \text{ if and only if} \\ &\qquad \exists R\, \exists X\, \big((R \subseteq P \times X) \wedge (X \subseteq V^{\mathbb{P}} \times \{``\in", ``="\} \times V^{\mathbb{P}}) \wedge \\ &\quad X \text{ is downwardly closed w.r.t. the appropriate well-founded relation} \wedge \\ &\qquad\qquad R \text{ satisfies the recursion defining } \Vdash^* \ \wedge\ (p, \sigma, e, \tau) \in R\big). \end{aligned}$$

On the other hand, if P is a proper class, this "translation" is not possible, since the initial approximations will still be class-sized, and hence not first-order quantifiable. For this reason, we assume more about P in order to guarantee that the forcing relation will be definable.

Definition 18. *Given a ZFC model M and an M-cardinal κ, an M-definable partial order P (on a subclass of M) is* pretame below κ *if for every M-definable sequence $\langle D_i \,|\, i \in \gamma\rangle$ of dense classes indexed by some $\gamma < \kappa$, and every $p \in P$, there is a $q \leq p$ and a sequence $\langle d_i \,|\, i \in \gamma\rangle \in M$ such that for each $i \in \gamma$, $d_i \subseteq D_i$ and d_i is predense $\leq q$. A partial order is* pretame *if it is pretame below κ for all cardinals $\kappa \in M$.*

Theorem 2.18 of [9] states that pretameness is sufficient to prove the definability of the forcing relation. We outline the proof here, showing that it can be used to prove a slightly more general statement that we will need in Section 3.6. We retain the class predicates in the statement of the theorem, as this level of generality will be needed when we come to Section 3.6. Of course, for ZFC models without such a predicate attached, the theorem can be made applicable by taking $A = \emptyset$, or indeed any other definable class. Henceforth, when we say that a model $\mathcal{M}$ with a class predicate satisfies ZFC, or any other axiom system including the axiom schemata of replacement and comprehension, we implicitly include that the model satisfies those schemata for formulae involving the predicate. Similarly, $\mathcal{M}$-definable will mean definable by a formula possibly involving the predicate.

Theorem 19. *Let $\mathcal{M} = \langle M, \in, A\rangle$ be a model of ZFC$-$ Power set, with class predicate $A \subseteq M$. Suppose there is an $\mathcal{M}$-definable, $\subseteq$-increasing sequence $\langle M_\alpha \,|\, \alpha \in Ord\rangle$ of sets in M such that $\mathcal{M} \models \forall x \exists \alpha (x \in M_\alpha)$ (for example, if $\mathcal{M} \models$ Power set, the von Neumann hierarchy suffices). Let P be an $\mathcal{M}$-definable, pretame class forcing. Then for any formula φ in the language of set theory extended with predicates for A, M, and the generic G, the relation $p \Vdash_P \varphi(\sigma_0, ..., \sigma_n)$ is $\mathcal{M}$-definable.*

Proof. As in the set forcing case, we wish to define a relation $\Vdash^*$, which we will later show to be equal to $\Vdash$, which has the following properties:

1. $p \Vdash^* \sigma \in \tau$ if and only if $\{q \,|\, \exists \langle \pi, r\rangle \in \tau (q \leq r \wedge q \Vdash^* \pi = \sigma)\}$ is dense $\leq p$, that is, for all $q \leq p$ there is an $r \leq q$ in this set.
2. $p \Vdash^* \sigma = \tau$ if and only if for all $\langle \pi, r\rangle \in \sigma \cup \tau$, $p \Vdash^* (\pi \in \sigma \leftrightarrow \pi \in \tau)$.
3. $p \Vdash^* \varphi \wedge \psi$ if and only if $p \Vdash^* \varphi$ and $p \Vdash^* \psi$.
4. $p \Vdash^* \neg\varphi$ if and only if for all $q \leq p$, $\neg(q \Vdash^* \varphi)$.
5. $p \Vdash^* \forall x \varphi(x)$ if and only if for all names σ, $p \Vdash^* \varphi(\sigma)$.
6. $p \Vdash^* M(\sigma)$ if and only if $\exists \alpha \in \mathrm{Ord}^{\mathcal{M}} (p \Vdash^* \sigma \in \check{M}_\alpha)$.

7. $p \Vdash^* A(\sigma)$ if and only if $\exists \alpha \in \mathrm{Ord}^{\mathcal{M}}(p \Vdash^* \sigma \in \check{(A \cap M_\alpha)})$.
8. $p \Vdash^* G(\sigma)$ if and only if $\exists \alpha \in \mathrm{Ord}^{\mathcal{M}}(p \Vdash^* \sigma \in \Gamma_\alpha)$, where

$$\Gamma_\alpha = \{\langle \check{p}, p\rangle \mid p \in P \cap M_\alpha\}.$$

For a set-sized partial order P, this list of properties is enough to give a recursive definition of the relation $\Vdash^* \varphi$ for any formula φ. Of course in the set forcing case, defining the predicate G is entirely unnecessary, and in most applications we do not need the other predicates M and A either.

On the other hand, if P is a class-sized partial order, then merely exhibiting these desired properties is not sufficient to give a first-order definition of a relation $\Vdash^*$ satisfying them, as was mentioned above. Thus, we will give another definition for $\Vdash^*$, justified by recursion on a well-founded relation, and show that it satisfies properties 1 to 8. Having defined $\Vdash^*$ in this way, it will then follow as in the set forcing case that $\Vdash^* = \Vdash$.

It suffices here to define $\Vdash^*$ for atomic formulae in such a way that properties 1 and 2 hold, because the definition for arbitrary formulae will then arise directly from properties 3–8 by induction on the structure of the formula, as it does in the set forcing case. It is convenient to also incorporate the definition of $\Vdash^*$ for negations of atomic formulae immediately, which we of course do according to property 4.

We define an intermediate function $F(p, \sigma, e, \tau)$ such that

$$F(p, \sigma, e, \tau) = (i, x),$$

where $i \in 2$ and x is a non-empty subset of $\{q \in P \mid q \leq p\}$, and claim that when $\Vdash^*$ is defined, we will have that

$$\begin{aligned} &\text{if } i = 0, \text{ then for all } q \in x,\ q \Vdash^* \neg(\sigma e \tau), \text{ and} \\ &\text{if } i = 1, \text{ then for all } q \in x,\ q \Vdash^* \sigma e \tau. \end{aligned} \qquad (\mathcal{F})$$

Having our hierarchy $\langle M_\alpha \mid \alpha \in \mathrm{Ord}^{\mathcal{M}}\rangle$ will allow us to make appropriate definitions for values x, so that for any p, σ, e and τ, the witnesses to the fact that $F(p, \sigma, e, \tau) = (i, x)$ can all be obtained within some M_α. To keep our notation tidy, we will also define $l(p, \sigma, e, \tau)$ to be the ordinal α indexing the appropriate M_α. Also notationally, note that Friedman in [9] generally uses d for the second component of F, where we use x; we have opted to reserve the variable d for sets that are predense below some condition q, to aid readability.

Assuming F is defined, we define the relation $\Vdash^*$ for atomic formulae and their negations as follows.

Definition 20. *For* $p \in P$, $\sigma, \tau \in M^P$, *and* $e \in \{$"$\in$", "$=$"$\}$, $p \Vdash^* \sigma e \tau$ *if and only if for all* $q \leq p$, *there is an* $r \leq q$ *and some* x *such that* $F(r, \sigma, e, \tau) = (1, x)$. *In line with property 4,* $p \Vdash^* \neg(\sigma e \tau)$ *if and only if for all* $q \leq p$, $\neg(q \Vdash^* \sigma e \tau)$.

Clearly this definition is downwardly closed. The verification that the relationship ($\mathcal{F}$) between F and $\Vdash^*$ indeed holds occurs during the construction of F by induction on (σ, e, τ), as also for the fact that $\Vdash^*$ satisfies properties 1, 2, and 4 for atomic formulae and their negations.

So we wish to define l and F by recursion on (σ, e, τ), with only set-many $q \in P$ to consider at each "step-down". For concreteness, let us set out an appropriate well-founded partial order (in the strict sense, with antisymmetry) on $M^P \times \{$"$\in$", "$=$"$\} \times M^P$, on which this recursion will take place.

Definition 21. *Suppose* $(\sigma, e, \tau), (\sigma', e', \tau') \in M^P \times \{$"$\in$", "$=$"$\} \times M^P$. *Say that* $(\sigma', e', \tau') < (\sigma, e, \tau)$ *if*

- $\max(\text{rank}(\sigma', \tau')) < \max(\text{rank}(\sigma, \tau))$, *or*
- $\max(\text{rank}(\sigma', \tau')) = \max(\text{rank}(\sigma, \tau))$, *and* $\text{rank}(\sigma) \geq \text{rank}(\tau)$ *but* $\text{rank}(\sigma') < \text{rank}(\tau')$, *or*
- $\max(\text{rank}(\sigma', \tau')) = \max(\text{rank}(\sigma, \tau))$, *and* $\text{rank}(\sigma) \geq \text{rank}(\tau) \leftrightarrow \text{rank}(\sigma') \geq \text{rank}(\tau')$, *and* $e =$ "$=$" *and* $e' =$ "$\in$".

Clearly $<$ is a well-founded relation on $M^P \times \{$"$\in$", "$=$"$\} \times M^P$, and it is straightforward to check that properties 1 and 2 always reduce the $<$-rank of the members of $M^P \times \{$"$\in$", "$=$"$\} \times M^P$ considered. Of course, this relation $<$ is somewhat *ad hoc*, especially since we rely on having "$\pi = \sigma$" rather than "$\sigma = \pi$" in property 1, but it does the job.

At last we are ready to give the recursive definitions of F and l. This proceeds exactly as is laid out in [9], Theorem 2.18. To aid the reader one final step further, we give here a formal definition for F and l in the case where $e =$ "$\in$" extracted from that reference, and leave the case of $e =$ "$=$" and the verification that it all works for the reader to obtain from [9].

Given p, σ and τ, for each $\alpha \in \text{Ord}$ let

$$A_\alpha = \bigcup \{x \in M_\alpha \mid \exists \langle \pi, r \rangle \in \tau \ \exists q \in M_\alpha \\ \big(q \leq p \wedge q \leq r \wedge F(q, \sigma, \text{“=”}, \pi) = (1, x)\big)\},$$

and

$$\begin{aligned} B_\alpha = \Big\{ q \in M_\alpha \,\Big|\, & \forall \langle \pi, r\rangle \in \tau \; \exists d \in M_\alpha \\ & \Big(\forall s \in d \,\big(s \perp r \;\vee\; \exists\, s' \in M_\alpha \,\exists\, x \\ & (s \leq s' \leq r \wedge F(s', \sigma, \text{“$=$”}, \pi) = (0, x) \wedge s \in x)\big) \\ & \qquad \wedge \forall t \leq q \,\exists s \in d \,\exists u \leq t (u \leq s)\Big) \Big\}. \end{aligned}$$

Let $l(p, \sigma, \text{“$\in$”}, \tau)$ denote the least α such that $A_\alpha \cup B_\alpha \neq \emptyset$ if such an α exists, and -1 otherwise. For $l(p, \sigma, \text{“$\in$”}, \tau) \neq -1$, let

$$F(p, \sigma, \text{“$\in$”}, \tau) = \begin{cases} (1, A_{l(p,\sigma,\text{“$\in$”},\tau)}) & \text{if } A_{l(p,\sigma,\text{“$\in$”},\tau)} \neq \emptyset \\ (0, B_{l(p,\sigma,\text{“$\in$”},\tau)}) & \text{otherwise.} \end{cases}$$

We of course claim that $l(p, \sigma, \text{“$\in$”}, \tau)$ is never -1; the verification of this, and indeed the rest of the proof, is left to the reader with reference to [9], Theorem 2.18. ⊣

With the forcing relation shown to be definable, the Truth Lemma and the preservation of ZFC$^-$ for pretame forcings now go through; the former is just as in the set forcing case, and for the remaining subtleties for the latter, the reader is referred to [9], Lemma 2.19.

We now progress to the requirements we impose on a class forcing that ensure that the Power Set axiom is also preserved.

Definition 22. *For M a model of ZFC, P a definable partial order on a subclass of M, and $p \in P$, a pair (D_0, D_1) of classes is a* predense $\leq p$ partition *if $D_0 \cup D_1$ is predense below p and $p_0 \in D_0$, $p_1 \in D_1$ implies that $p_0 \perp p_1$. Two sequences $\langle (D^i_0, D^i_1) \,|\, i \in a\rangle$ and $\langle (E^i_0, E^i_1) \,|\, i \in a\rangle$ of predense $\leq p$ partitions are* equivalent $\leq p$ *if for each $i \in a$,*

$$\{q \in P \,|\, \exists r \in D^i_0 (q \leq r) \leftrightarrow \exists r \in E^i_0 (q \leq r)\} \textit{ is dense} \leq p.$$

In each case we omit the “$\leq p$” if $p = 1_P$.

Definition 23. *For M and κ as in Definition 18 above, and P an M-definable partial order, P is* tame *below κ if P is pretame below κ, and for each $\gamma < \kappa$ and $p \in P$ there is a $q \leq p$ and an $\alpha \in Ord(M)$ such that for each sequence $\mathbf{D} = \langle (D^i_0, D^i_1) \,|\, i \in \gamma\rangle \in M$ of predense $\leq q$ partitions,*

$$\{r \in P \,|\, \exists \mathbf{E} = \langle (E^i_0, E^i_1) \,|\, i \in \gamma\rangle \in V^M_\alpha (\mathbf{D} \textit{ is equivalent} \leq r \textit{ to } \mathbf{E})\}$$

is dense below q. The partial order P is tame *if it is tame below κ for all cardinals $\kappa \in M$.*

If P is tame, then a generic extension $\langle M[G], M, G\rangle$ of M by P will satisfy ZFC, and conversely - these are Theorem 2.21 and Proposition 2.20 of [9]. Also, P is pretame if and only if $\langle M[G], M, G\rangle$ is always a model of ZF – Power set — the reverse direction is Proposition 2.17 of [9]. In each case we are making the usual assumption that a generic exists through every condition of the partial order.

A fact that will be useful to us is that $M[G]$ will satisfy replacement with respect to formulae with predicates for M and G (see Lemma 2.19 of [9]), so little generality is lost by allowing such predicates. On the other hand, note that in Chapter 4 we will be constructing an $M[G]$ with a well-order definable from $M[G]$ alone, without the aid of such class predicates.

Our class forcings will be reverse Easton iterations of set forcings. To fix terminology, we formally define this here, at a level of generality sufficient for our purposes but which is restricted enough to simplify some definitions. In particular, the partial order Q_α with respect to which we force at level α will be definable in $V[G_\alpha]$, so by identifying Q_α with its definition we can talk about Q_α without having constructed $V[G_\alpha]$ or given a name $\dot{Q}_\alpha$ for it. Of course, we then define and use a canonical name $\dot{Q}_\alpha$ for Q_α

Suppose that $x \in M$, $P \in M$ is a set forcing, G is P-generic over M, and $y \in M[G]$ is built from x using pairing, union, power set and replacement. Then there is a canonical name for y, using $\check{x}$ and the explicit constructions used to prove those axioms in $M[G]$. Also recall that given a name $\dot{Q}$ for a partial order, there is a set $E_{\dot{Q}}$ of names such that

1. for every $\sigma \in E_{\dot{Q}}$, $1_P \Vdash \sigma \in \dot{Q}$, and
2. for every P-name ρ there is a $\sigma \in E_{\dot{Q}}$ such that

$$1_P \Vdash \rho \in \dot{Q} \to \rho = \sigma.$$

In particular, note that we can (and do) take $E_{\dot{Q}}$ to be a set and not a proper class. By a "name for an element of Q" we shall mean an element of this set $E_{\dot{Q}}$.

Definition 24. *Suppose that for each $\alpha \in Ord$ we have Q_α a set forcing partial order (consisting of the domain, the order, and a maximum element 1_{Q_α}) definable from α using the pairing, union, power set and replacement axioms. Then we say that P is the reverse Easton iteration of Q_α if there are sequences $\langle P_\alpha \,|\, \alpha \in Ord\rangle$ and $\langle \dot{Q}_\alpha \,|\, \alpha \in Ord\rangle$ such that the following requirements hold, and P is the direct limit of the sequence $\langle P_\alpha \,|\, \alpha \in Ord\rangle$.*

i. *Each* P_α *is a partial order, and each* $\dot{Q}_\alpha$ *is the canonical* P_α*-name for* Q_α*. We further denote the canonical name for the maximum element by* $\dot{1}_{Q_\alpha}$*, so that in particular*

$$1_{P_\alpha} \Vdash (\dot{Q}_\alpha \textit{ is a partial order}) \wedge \forall q \in \dot{Q}_\alpha (q \mathrel{\dot{\leq}_{Q_\alpha}} \dot{1}_{Q_\alpha}).$$

ii. *For every ordinal* α*, every element of* P_α *is a* α*-tuple, and* P_0 *is the trivial forcing* $\{\emptyset\}$*.*

iii. *Given* $\alpha \in$ *Ord,* $p \in P_\alpha$*, and* $\gamma < \alpha$*, we have that* $p \restriction \gamma \in P_\gamma$ *and* $p(\gamma)$ *is a* P_γ*-name for an element of* Q_γ*.*

iv. *For every ordinal* α*,* $P_{\alpha+1}$ *is the set of all* $\alpha+1$*-tuples* p *satisfying (iii). For* p *and* q *in* $P_{\alpha+1}$*,* $p \leq q$ *if and only if* $p \restriction \alpha \leq_{P_\alpha} q \restriction \alpha$ *and*

$$p \restriction \alpha \Vdash p(\alpha) \mathrel{\dot{\leq}_{Q_\alpha}} q(\alpha).$$

v. *For each limit ordinal* α *and all* p *and* q *in* P_α*,* $p \leq_{P_\alpha} q$ *if and only if for all* $\gamma < \alpha$*,* $p \restriction \gamma \leq_{P_\gamma} q \restriction \gamma$*.*

vi. *For* α *a regular cardinal,* P_α *is the set of all* α*-tuples* p *satisfying (iii) such that there is a* $\beta < \alpha$ *with* $p(\gamma) = \dot{1}_{Q_\gamma}$ *for all* $\gamma > \beta$ *(that is, direct limits are taken at regular cardinal stages).*

vii. *For* α *a singular limit ordinal,* P_α *is the set of all* α*-tuples* p *satisfying (iii) (that is, inverse limits are taken at singular limit stages).*

Of course, we take $\langle \dot{1}_{Q_\beta} \mid \beta < \alpha \rangle$ to be the canonical maximum element of P_α, and for $\gamma < \alpha$ we have the complete embedding $i_{\gamma,\alpha} : P_\gamma \hookrightarrow P_\alpha$ given by $i_{\gamma,\alpha}(p) = p^\frown(\langle \dot{1}_{Q_\beta} \mid \gamma \leq \beta < \alpha \rangle)$. It is with respect to these embeddings that we take P to be the direct limit of $\langle P_\alpha \mid \alpha \in \mathrm{Ord} \rangle$. For notational convenience we may sometimes write $P = P_\alpha$ with $\alpha = \mathrm{Ord}$.

If the forcing poset Q_α is only nontrivial at those stages α of the form $\beth_\beta$ for some ordinal β (if the GCH holds, that means for infinite cardinals α) then taking inverse limits everywhere except at regular cardinal stages is the same as taking inverse limits everywhere except at inaccessible cardinal stages. Considering the iteration P' having $Q_{\beth_\beta}$ at stage β, taking direct limits at strongly inaccessible β and indirect limits elsewhere, we see by induction on β that P'_β is isomorphic to $P_{\beth_\beta}$ (assuming $E_{\dot{Q}_{\beth_\beta}}$ is canonically chosen relative to $P_{\beth_\beta}$ and P'_β respectively, so that the isomorphism extends to names for elements of $Q_{\beth_\beta}$). Thus, it is reasonable to also refer to iterations which only take direct limits at inaccessible stages and indirect limits elsewhere as reverse Easton iterations. Our forcing to obtain the GCH will be of this latter form, and those to obtain morasses and to obtain a definable well-order of the universe will be of the "sparse" form where it makes no difference.

A key fact about iterations similar to reverse Easton iterations is the following; see for example Lemma 21.8 of [11].

Lemma 25 (The Factor Lemma). *Let P be an iteration (of set or class length) of set forcings Q_α such that only direct and inverse limits are taken at limit stages. If α is such that for every limit $\xi > \alpha$ with cf(ξ) $\leq |P_\alpha|$ the inverse limit is taken at stage ξ, then P may be factorised as $P \cong P_\alpha * \dot{P}^\alpha$, where $\dot{P}^\alpha$ names the iteration in $V[G_\alpha]$ of the Q_β, $\beta \geq \alpha$, with supports as in P.*

Of course, when we are only taking direct limits at inaccessible stages in our reverse Easton iteration, this lemma shows that the iteration can be factored at any point. We shall use this fact without mention throughout.

In section 2.3 of [9] it is shown that a particular reverse Easton iteration over L is tame, but it is clear that the same argument lifts to give the follow result.

Proposition 26. *Let P be a reverse Easton iteration of set forcings Q_α, such that for every regular cardinal κ there is an α_κ such that*

$$P_{\alpha_\kappa} \Vdash \dot{P}^{\alpha_\kappa} \textit{ is } \kappa\textit{-distributive.}$$

Then P is tame.

Proof. (Outline) By Lemma 2.22 of [9], κ-distributivity implies tameness below κ. Hence, we have

$$P_{\alpha_\kappa} \Vdash \dot{P}^{\alpha_\kappa} \text{ is tame below } \kappa.$$

For each κ, P_{α_κ} is a set forcing, and hence tame. Since P_{α_κ} is tame and forces that $\dot{P}^{\alpha_\kappa}$ is pretame below κ, it follows from Lemma 2.31 of [9] that $P \cong P_{\alpha_\kappa} * \dot{P}^{\alpha_\kappa}$ is pretame below κ. This holds for all κ, and so P is pretame, whence each $\dot{P}^\alpha$ is forced to be pretame, by the $M[G] \models \mathrm{ZF} - \text{Power set}$ characterisation. Now for every κ, we have that P_{α_κ} is tame and forces that $\dot{P}^{\alpha_\kappa}$ is tame below κ and pretame, and so it follows from Lemma 2.31 of [9] again that $P \cong P_{\alpha_\kappa} * \dot{P}^{\alpha_\kappa}$ is tame below κ. Since this is true for all α, P is tame. ⊣

1.4 Lifting Embeddings

To achieve our results regarding the preservation of large cardinals while performing our forcing constructions, we use embedding characterisations of the large cardinals, and show that the embeddings may be "lifted" appropriately. The following key lemma is due to Silver.

Lemma 27 (The Lifting Lemma). *Let $j : V \to M$ be an elementary embedding, and let P be a definable, pretame forcing. Let G be P^V-generic over V and G' be P^M-generic over M. If $j``G \subseteq G'$, then $j^* : V[G] \to M[G']$ defined by*

$$j^*(\sigma^G) = (j(\sigma))^{G'}$$

is a (well-defined) elementary embedding.

Proof. Let $\sigma_0, \ldots, \sigma_{n-1}$ be P^V-names in V, let $\varphi(x_0, \ldots, x_{n-1})$ be a formula in n variables, and suppose that $V[G] \vDash \varphi(\sigma_0^G, \ldots, \sigma_{n-1}^G)$. Note that this covers the case in which $V[G] \vDash \sigma_0^G = \sigma_1^G$, so well-definedness will be proven alongside elementarity. By the Truth Lemma, there is some $p \in G$ such that $p \Vdash \varphi(\sigma_0, \ldots, \sigma_{n-1})$. Since the relation $\Vdash \varphi$ is first-order definable, it follows that $j(p) \Vdash \varphi(j(\sigma_0), \ldots, \sigma_{n-1})$. Now if $j``G \subseteq G'$, then $j(p) \in G'$, and so $M[G'] \vDash \varphi(j(\sigma_0)^{G'}, \ldots, j(\sigma_{n-1})^{G'})$, as required. ⊣

Of course, many large cardinal axioms generally require M to be an inner model of V, so to lift those large cardinals it will be necessary to have G' be a subclass of $V[G]$. Also, most large cardinal axioms assert more about the elementary embedding than simply the fact that it exists, so these further requirements will also need to be independently checked in each case.

In many cases, the requirement that $j``G$ be a subset of G' may be shown to be equivalent to requiring a particular condition to lie in G'. Such a condition is referred to as *master condition*.

Chapter 2

GCH

One of the first combinatorial principles one would like to force while preserving various large cardinals is the *GCH*, the generalised continuum hypothesis, which states that for every cardinal κ, $2^\kappa = \kappa^+$. The iterated forcing we will use to obtain the GCH is a fairly standard, natural one; the main issue will be to show that forcing with this particular partial order does preserve the large cardinals we are interested in. In many cases, this was achieved in [8]; we show that similar arguments go through for 1-extendible cardinals. This will also be helpful when we come to force the existence of morasses in the presence of 1-extendibles in chapter 3.

2.1 Forcing the GCH

We use the same forcing as in [8], performing a reverse Easton iteration of Cohen forcings, at each stage collapsing the old $2^{\aleph_\alpha}$ to $\aleph_{\alpha+1}$. Although this is a known forcing, we will be careful in our presentation, in order to set the stage for the theorems about this forcing that are to follow.

We define the class-sized iteration $\langle P_\alpha \,|\, \alpha \in \mathrm{Ord}\rangle$ of $\langle \dot{Q}_\alpha \,|\, \alpha \in \mathrm{Ord}\rangle$ by transfinite recursion. At stage α, having already forced the GCH to hold below $\aleph_\alpha$, we wish to force so as to collapse $2^{\aleph_\alpha}$ to $\aleph_{\alpha+1}$. The obvious choice for the forcing at this stage, then, is $\mathrm{Fn}(\aleph_{\alpha+1}, 2^{\aleph_\alpha}, \aleph_{\alpha+1})$. Note that $\mathrm{Fn}(\aleph_{\alpha+1}, 2, \aleph_{\alpha+1})$ is an equivalent forcing: letting π_1 denote the projection onto the first component of a product, we have

$$
\begin{aligned}
\mathrm{Fn}(\aleph_{\alpha+1}, 2^{\aleph_\alpha}, \aleph_{\alpha+1}) &\cong \mathrm{Fn}(\aleph_{\alpha+1}, {}^{<\aleph_{\alpha+1}}2, \aleph_{\alpha+1}) \\
&\overset{\text{dense}}{\hookleftarrow} \{p \in \mathrm{Fn}(\aleph_{\alpha+1}, {}^{<\aleph_{\alpha+1}}2, \aleph_{\alpha+1}) \mid \mathrm{dom}(p) \in \mathrm{Ord}\} \\
&\cong \big\{q \in \mathrm{Fn}(\aleph_{\alpha+1} \times \aleph_{\alpha+1}, 2, \aleph_{\alpha+1}) \mid \\
&\qquad\quad \pi_1(\mathrm{dom}(q)) \in \mathrm{Ord} \wedge \forall \beta < \pi_1(\mathrm{dom}(q)) \\
&\qquad\qquad\qquad \big(\{\gamma \mid \langle \beta, \gamma \rangle \in \mathrm{dom}(q)\} \in \mathrm{Ord}\big)\big\} \\
&\overset{\text{dense}}{\hookrightarrow} \mathrm{Fn}(\aleph_{\alpha+1} \times \aleph_{\alpha+1}, 2, \aleph_{\alpha+1}) \\
&\cong \mathrm{Fn}(\aleph_{\alpha+1}, 2, \aleph_{\alpha+1})
\end{aligned}
$$

where of course the central isomorphism between the two dense sets takes any p to the function $\langle \beta, \gamma \rangle \mapsto p(\beta)(\gamma)$. This is Exercise (VII.G3) in [14].

We therefore define the GCH partial order as follows.

Definition 28. *The* GCH Partial Order P *is the reverse Easton iteration of* $\langle \dot{Q}_\alpha \mid \alpha \in \mathit{Ord} \rangle$*, in which direct limits are taken only at strongly inaccessible cardinals and inverse limits are taken at other limit stages, and where* $\dot{Q}_\alpha$ *is the canonical* P_α*-name for* $\mathrm{Fn}(\aleph_{\alpha+1}, 2, \aleph_{\alpha+1})$*, with* $\aleph_{\alpha+1}$ *to be computed in* $V[G_\alpha]$*.*

For the rest of this chapter $P = P_{\mathrm{Ord}}$ will denote the GCH Partial Order, with P_α denoting the iteration after α stages, $G = G_{\mathrm{Ord}}$ denoting a P-generic over V, G_α denoting $G \restriction P_\alpha$, and Q_α denoting $\mathrm{Fn}(\aleph_{\alpha+1}, 2, \aleph_{\alpha+1})$ in $V[G_\alpha]$.

Before proceeding with the analysis of this iteration, let us recall the basic properties of the forcing poset $\mathrm{Fn}(\aleph_{\alpha+1}, 2, \aleph_{\alpha+1})$. It is $\aleph_{\alpha+1}$-closed, and by Lemma 2 (or simply considering its cardinality) we see that it is $(2^{\aleph_\alpha})^+$-cc, so the only cardinals collapsed by $\mathrm{Fn}(\aleph_{\alpha+1}, 2, \aleph_{\alpha+1})$ are those κ such that $\aleph_{\alpha+1} < \kappa \leq 2^{\aleph_\alpha}$. Of course, since $\mathrm{Fn}(\aleph_{\alpha+1}, 2, \aleph_{\alpha+1})$ is equivalent to $\mathrm{Fn}(\aleph_{\alpha+1}, 2^{\aleph_\alpha}, \aleph_{\alpha+1})$, a surjection $\aleph_{\alpha+1} \to 2^{\aleph_\alpha}$ is added by the forcing, and all such κ are indeed collapsed. The cardinality of $\mathrm{Fn}(\aleph_{\alpha+1}, 2, \aleph_{\alpha+1})$ is $2^{\aleph_\alpha}$, so there are at most $2^{2^{\aleph_\alpha}}$ antichains of $\mathrm{Fn}(\aleph_{\alpha+1}, 2, \aleph_{\alpha+1})$, and a nice names argument gives that the continuum function is unchanged at and above the ground model $2^{\aleph_\alpha}$ by forcing with $\mathrm{Fn}(\aleph_{\alpha+1}, 2, \aleph_{\alpha+1})$. Similarly, $2^{\aleph_{\alpha+1}}$ as computed in the extension will be the ground model $2^{2^{\aleph_\alpha}}$. Of course, $\aleph_{\alpha+1}$-closure implies that no new subset of $\aleph_\alpha$ is added, so the continuum function is unchanged below $\aleph_\alpha$, and after the forcing we have $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

Now, to the GCH partial order itself.

Lemma 29. *The GCH Partial Order is tame.*

Proof. Note that by the Factor Lemma, P may be written as $P_\alpha * \dot{P}^\alpha$, where $\dot{P}^\alpha$ names the iteration starting with Q_α. Each iterand in $\dot{P}^\alpha$ is forced to be $\aleph_{\alpha+1}$-closed, and inverse limits are taken everywhere except at inaccessibles of V, which will remain inaccessible after forcing with P_α. Indeed, it can be shown by induction that P_α has a dense suborder of size at most $\beth^V_{\alpha+1}$; but proving that here would take us too far afield.

We therefore have that

$$1_{P_\alpha} \Vdash \dot{P}^\alpha \text{ is } \aleph_{\check{\alpha}+1}\text{-closed},$$

so tameness is immediate from Proposition 26. ⊣

Lemma 30. *Let α be an ordinal or Ord. Then for every ordinal $\gamma \leq \alpha$, $\beth_\gamma^{V[G_\alpha]} = \aleph_\gamma^{V[G_\alpha]}$.*

Proof. As in the proof of Lemma 29 above, $P_\alpha$may be factorised as $P_\gamma * \dot{P}^{[\gamma,\alpha)}$, where $\dot{P}^{[\gamma,\alpha)}$ names the iteration starting with Q_γ, and we have

$$1_{P_\gamma} \Vdash \dot{P}^{[\gamma,\alpha)} \text{ is } \aleph_{\check{\gamma}+1}\text{-closed}.$$

It hence suffices to show that $\beth_\gamma^{V[G_\gamma]} = \aleph_\gamma^{V[G_\gamma]}$. for every γ. We do this by induction.

The case $\gamma = 0$ is trivial, and for γ a limit ordinal, we have $\beth_\beta^{V[G_\gamma]} = \aleph_\beta^{V[G_\gamma]}$ for all $\beta < \gamma$, whence $\beth_\gamma^{V[G_\gamma]} = \aleph_\gamma^{V[G_\gamma]}$. If γ is a successor ordinal, say $\gamma = \beta + 1$, then $P_\gamma \cong P_\beta * \mathrm{Fn}(\aleph_{\beta+1}, 2, \aleph_{\beta+1})$, and by the inductive hypothesis, $\beth_\beta^{V[G_\beta]} = \aleph_\beta^{V[G_\beta]}$. It then follows from the discussion of $\mathrm{Fn}(\aleph_{\beta+1}, 2, \aleph_{\beta+1})$ above that $\beth_{\beta+1}^{V[G_{\beta+1}]} = \aleph_{\beta+1}^{V[G_{\beta+1}]}$, as required. ⊣

In particular, these two lemmas combine to show that forcing with P yields a model of ZFC + GCH.

2.2 1-extendible cardinals and the GCH

In [8], Friedman shows that when forcing with the GCH forcing P, any given hyperstrong or n-superstrong cardinal may be preserved. In fact, the same proof essentially gives the following stronger statement.

Theorem 31 (S. Friedman). *Forcing with the GCH partial order P preserves all hyperstrong cardinals and n-superstrong cardinals, for all $n \in \omega + 1$.*

Proof. Observe that each iterand in the GCH forcing is very homogeneous: given a condition $p \in Q_\alpha = \mathrm{Fn}(\aleph_{\alpha+1}, 2, \aleph_{\alpha+1})$ and a Q_α-generic g, there is an automorphism π of Q_α such that $p \in \pi``g$. Moreover, we have that the partial orders Q_α for later stages will not be changed (modulo forced equality) by automorphisms for initial segments of the iteration. It is known that in these circumstances the homogeneity of the individual partial orders carries across to the entire iteration; see [7] for the details. It follows that all the arguments for Theorem 2 of [8], which in particular choose P-generics containing a certain master condition for each large cardinal κ, can be carried out within $V[G]$. Therefore, every large cardinal of the listed kinds is preserved in $V[G]$. $\dashv$

We here show that the same result is true for 1-extendible cardinals.

Theorem 32. *For any model V of ZFC, there is a class-generic extension $V[G]$ of V such that $V[G] \vDash$ GCH, and in which every 1-extendible cardinal of V remains 1-extendible.*

Proof. We shall of course force with the GCH forcing P. Let κ be 1-extendible in V, let $j : H_{\kappa^+} \to H_{\lambda^+}$ in V witness this fact, and let G be P-generic over V. We shall show that j may be lifted to an embedding $j^* : H_{\kappa^+}^{V[G]} \to H_{\lambda^+}^{V[G]}$ witnessing the 1-extendibility of κ in $V[G]$.

The GCH forcing P may be factorised as $P_\kappa * P^\kappa$, with P^κ a κ^+-closed forcing. Hence, $H_{\kappa^+}^{V[G]} = H_{\kappa^+}^{V[G_\kappa]}$. Similarly, $H_{\lambda^+}^{V[G]} = H_{\lambda^+}^{V[G_\lambda]}$. Now because κ is inaccessible, P_κ has cardinality κ and lies in H_{κ^+}. Recall that forcing over a model of ZFC$^-$ yields a model of ZFC$^-$, and note that G_κ is generic over H_{κ^+} for the set forcing P_κ. We claim that in fact $H_{\kappa^+}^{V[G_\kappa]} = H_{\kappa^+}^V[G_\kappa]$, the generic extension of the model $H_{\kappa^+}^V$ of ZFC$^-$ by G_κ. For this we need to show that if σ is a P_κ-name in V such that $\sigma_{G_\kappa} \in H_{\kappa^+}^{V[G_\kappa]}$, then there is a name $\tau \in H_{\kappa^+}^{P_\kappa}$ such that $\sigma_{G_\kappa} = \tau_{G_\kappa}$. But now every element of H_{κ^+} can be obtained as the Mostowski collapse of a relation on κ given by a subset of $\kappa \times \kappa$ (and this process does not appeal to the Power Set Axiom). Every subset of $\kappa \times \kappa$ in the extension $V[G_\kappa]$ has a nice name of the form

$$\bigcup_{(\alpha,\beta)\in\kappa\times\kappa} \{\check{(\alpha,\beta)}\} \times A_{(\alpha,\beta)}$$

where each $A_{(\alpha,\beta)}$ is an antichain in P_κ. Since $|P_\kappa| = \kappa$, such nice names lie in $H_{\kappa^+}^V$. Therefore, every subset of $\kappa \times \kappa$ in $V[G_\kappa]$ is also in $H_{\kappa^+}[G_\kappa]$, and consequently we indeed have that $H_{\kappa^+}^{V[G_\kappa]} = H_{\kappa^+}^V[G_\kappa]$. Of course by elementarity, λ is also inaccessible, and so also $H_{\lambda^+}^{V[G_\lambda]} = H_{\lambda^+}^V[G_\lambda]$.

We are thus reduced to showing that $j : H_{\kappa^+} \to H_{\lambda^+}$ may be lifted to a $j^* : H_{\kappa^+}[G_\kappa] \to H_{\lambda^+}[G_\lambda]$. We wish to apply the Lifting Lemma (Lemma 27), and must show that $j``G_\kappa \subseteq G_\lambda$. But now $j \restriction P_\kappa$ is the identity function, so $j``G_\kappa = G_\kappa \subset G_\lambda$. Therefore the Lifting Lemma gives us that j lifts to $j^* : H_{\kappa^+}^{V[G]} \to H_{\lambda^+}^{V[G]}$, and so κ is 1-extendible in $V[G]$. $\dashv$

In Section 3.6, we will show that 1-extendible cardinals κ may be preserved while forcing morasses to exist. To show that the forcing is pretame from the perspective of H_κ^+, we will first force GCH as above, and use the fact that after our GCH forcing, $V[G] = L[G]$, and further $H_{\kappa^+}^{V[G]} = L_{\kappa^+}[A]$, where A is taken to be a class predicate over H_{κ^+} for the Cohen set added at stage κ. This gives a stratification of H_{κ^+} as is required for Theorem 19, from which we will be able to deduce that the forcing relation for our mangrove forcing is definable.

However, $p \Vdash \varphi(\sigma_0, \ldots, \sigma_{n-1})$ will only be definable relative to A, and so the Lifting Lemma will not suffice to lift the embedding j witnessing 1-extendibility, as j need not respect arbitrary class predicates. We show here that in fact j *does* respect the predicate A given by the generic, giving a mild strengthening of Theorem 32. The crux of the proof will be the definability of the forcing relation for $P_{\kappa+1}$, which from the point of view of H_{κ^+} is a class forcing. Theorem 19 is not applicable, since we don't yet have a stratification of H_{κ^+} into sets. However, we will be able to demonstrate the definability of forcing for $P_{\kappa+1}$ anyway, because it is such a well-behaved forcing partial order.

Definition 33. *The language $\mathcal{L}_{STG}$ is the language obtained from the language of set theory by adding a single unary predicate G.*

Lemma 34. *Let $\kappa \in M$ be an inaccessible cardinal, and let Q_κ denote* $\mathrm{Fn}(\kappa^+, 2, \kappa^+)$*, the forcing at stage κ in the GCH forcing above. Then the forcing relation $\Vdash \varphi$ for forcing over H_{κ^+} with Q_κ is (uniformly) definable over H_{κ^+} for $\mathcal{L}_{STG}$ formulae φ, where the predicate G is to be interpreted as the Q_κ-generic.*

Proof. As mentioned above, the distinction between the situation of this proposition and that of Theorem 19 is the lack of a stratification of M by sets. This renders the method of proof used for that theorem unviable, and we must rely on the closure properties of Q_κ to demonstrate the definability of the forcing relation for atomic formulae. We also need to entirely restate Property 8 for $\Vdash^*$ as given in Theorem 19, as it too makes use of the stratification of M. Dealing with this first, let us define $p \Vdash^* G(\sigma)$ to mean

$$\forall q \leq p\, \exists r \leq q\, \exists s \geq r (r \Vdash^* \sigma = \check{s}), \tag{8'}$$

that is, it is dense below p to force σ to be something specific greater than yourself. If $p \in G$, then by genericity there is some $r \in G$ and some s greater than or equal to r and consequently also in G such that $r \Vdash^* \sigma = \check{s}$, whence $\sigma_G \in G$. Conversely, suppose $\sigma_G \in G$ for every $G \ni p$; let $q \leq p$ and fix some $G \ni q$. Once we have shown that the forcing relation is definable for atomic formulae, it will follow from the truth lemma there is some $r' \in G$ such that

$$r' \Vdash \sigma = \check{(\sigma_G)}.$$

Taking $r \in G$ such that $r \leq q$, $r \leq r'$, and $r \leq \sigma_G$, we have that r makes (8′) hold with σ_G as s. Hence, (8′) is indeed a formal definition encapsulating the statement that for every generic containing p, σ_G is in G, and it is clear that the argument extends to prove the Truth Lemma valid for formulae involving the predicate G.

With forcing the predicate G so redefined, we can move on to show that the forcing relation is indeed definable. Let σ and τ be Q_κ-names in H_{κ^+}, and let

$$R_{\sigma,\tau} = \mathrm{trcl}(\{\sigma_0, \ldots, \sigma_n\}) \cap P,$$

the set of conditions hereditarily appearing in either σ or τ. Note that since the names σ and τ are in H_{κ^+}, $|R_{\sigma,\tau}| \leq \kappa$. Let $d_{\sigma,\tau} = \bigcup_{r \in R_{\sigma,\tau}} \mathrm{dom}(r)$; then $d_{\sigma,\tau}$ is a subset of κ^+ of size at most κ.

Now, for conditions q such that $d_{\sigma,\sigma} \subseteq \mathrm{dom}(q)$, we may recursively define the evaluation of σ at q by

$$\mathrm{val}(\sigma, q) = \{\mathrm{val}(\rho, q) \mid \exists X \subseteq \mathrm{dom}(q)(\langle \rho, q \restriction X \rangle \in \sigma)\}.$$

Observe that if G is any Q_κ-generic over H_{κ^+} containing such a q, then $\sigma_G = \mathrm{val}(\sigma, q)$. Conversely, for any G Q_κ-generic over H_{κ^+} and element q of G with domain containing $d_{\sigma,\sigma}$, we have $\mathrm{val}(\sigma, q) = \sigma_G$. Thus, the evaluation of any name in the generic extension may be entirely determined by a single condition.

With this in hand, we may define

$$p \Vdash^* \sigma \in \tau \quad \longleftrightarrow \quad \forall q \leq p\big(\mathrm{dom}(q) \supseteq d_{\sigma,\tau} \to \mathrm{val}(\sigma, q) \in \mathrm{val}(\tau, q)\big)$$

and

$$p \Vdash^* \sigma = \tau \quad \longleftrightarrow \quad \forall q \leq p\big(\mathrm{dom}(q) \supseteq d_{\sigma,\tau} \to \mathrm{val}(\sigma, q) = \mathrm{val}(\tau, q)\big).$$

Then clearly $p \Vdash^* \sigma \in \tau$ if and only if $\sigma_G \in \tau_G$ for every Q_κ-generic $G \ni p$, and $p \Vdash^* \sigma = \tau$ if and only if $\sigma_G = \tau_G$ for every Q_κ-generic $G \ni p$. $\dashv$

Of course having shown that the forcing relation is definable, we will also want the Truth Lemma to hold, and indeed the argument for it goes through without any difficulty.

Theorem 35. *Let V be a model of ZFC and let κ be a 1-extendible cardinal in V with 1-extendibility witnessed by an elementary embedding $j : H_{\kappa^+} \to H_{\lambda^+}$. Let $V[G]$ be a P-generic extension of V. Then there is a $G' \subset V[G]$ which is P-generic over V such that $V[G] = V[G']$ and the lift j^* of j to $H_{\kappa^+}^{V[G']}$ (as in the proof of Theorem 32) is elementary between the $\mathcal{L}_{STG}$-structures $\langle H_{\kappa^+}^{V[G']}, G'(\kappa)\rangle$ and $\langle H_{\lambda^+}^{V[G']}, G'(\lambda)\rangle$.*

Proof. In Theorem 32 the lift j^* was constructed in $H_{\kappa^+}^{V[G_\kappa]}$, after observing that P^κ is κ^+-closed and hence does not affect H_{κ^+}. We now claim that the P^κ generic G^κ may be chosen so that this same j^* is also elementary for formulae in the language $\mathcal{L}_{STG}$. The proof is essentially as for Lemma 27. By the Truth Lemma, every $\mathcal{L}_{STG}$ sentence φ true in $\langle H_{\kappa^+}^{V[G_{\kappa+1}]}, G(\kappa)\rangle = \langle H_{\kappa^+}^{V[G]}, G(\kappa)\rangle$ is forced (over $H_{\kappa^+}^{V[G_\kappa]}$) to be true by some $p \in Q_\kappa$. Now by the definability of the forcing relation, we have

$$p \Vdash_{Q_\kappa} \varphi(\sigma_0, \dots, \sigma_n) \leftrightarrow j^*(p) \Vdash_{Q_\lambda} \varphi(j^*(\sigma_0), \dots, j^*(\sigma_n)).$$

Therefore, if $j^*``G(\kappa) \subseteq G(\lambda)$, we will have that $j^*$ is elementary from $\langle H_{\kappa^+}^{V[G]}, G(\kappa)\rangle$ to $\langle H_{\lambda^+}^{V[G]}, G(\lambda)\rangle$. But now $|G(\kappa)|^{V[G_\lambda]} = (\kappa^+)^{V[G_\lambda]} < \lambda$, so $j^*``G(\kappa)$ is a pairwise-compatible set of conditions in $\mathrm{Fn}(\lambda, 2, \lambda)^{V[G_\lambda]}$ (indeed, in $\mathrm{Fn}(\lambda, 2, \lambda)^{V[G_\kappa]}$) of size less than λ. Hence, $m = \bigcup j^*``G(\kappa)$ is a single condition in $\mathrm{Fn}(\lambda, 2, \lambda) = Q_\lambda$, and so if $G$ is chosen such that this master condition lies in $G(\lambda)$, then we will indeed have $j^*``G(\kappa) \subset G(\lambda)$. This is easy to arrange by modifying $G(\lambda)$ to obtain $G'(\lambda)$. Let q be that element of $G(\lambda)$ with $\mathrm{dom}(q) = \mathrm{dom}(m)$, and define $\psi : Q_\lambda \to Q_\lambda$ by

$$\psi(f(\alpha)) = \begin{cases} m(\alpha) & \text{if } f(\alpha) = q(\alpha) \\ q(\alpha) & \text{if } f(\alpha) = m(\alpha) \\ f(\alpha) & \text{otherwise} \end{cases}$$

for all $f \in Q_\lambda$ and $\alpha \in \mathrm{dom}(f)$. Clearly ψ is a (self-inverse) automorphism of Q_λ in $V[G]$, whence $G'(\lambda) = \psi``G(\lambda)$ is Q_λ-generic over $V[G_\lambda]$, and $V[G_{\lambda+1}] = V[G_\lambda * G'(\lambda)]$. The "tail" generic $G^{\lambda+1}$ will therefore still be $P^{\lambda+1}$-generic over $V[G_\lambda * G'(\lambda)]$. Hence, considering P as $P_\lambda * Q_\lambda * P^{\lambda+1}$, we may take $G' = G_\lambda * G'(\lambda) * G^{\lambda+1}$, which is P-generic over V and satisfies $V[G] = V[G']$. Moreover, $G'(\lambda)$ contains the master condition m determined by $G(\kappa) = G'(\kappa)$, so $j^*``G'(\kappa) \subset G'(\lambda)$, and so j^* is elementary from $\langle H_{\kappa^+}^{V[G']}, G'(\kappa)\rangle$ to $\langle H_{\lambda^+}^{V[G']}, G'(\lambda)\rangle$, as required. ⊣

Chapter 3

Morasses and Mangroves

Morasses were formulated by Jensen to abstract out properties of L useful for proving cardinal transfer theorems. Their existence has since been shown by Velleman [17] to be equivalent to the existence of *simplified morasses*, but while these latter structures are much more manageable, they can essentially only be obtained in L via morasses. In keeping with the theme of this thesis, we therefore stick with morasses, particularly the variety we call *mangroves.*

3.1 What they are

In this section we follow fairly closely the excellent presentation in Velleman's paper [16]. In particular, we retain the notation M.1–M.7 for axioms of a morass, used both there and in Devlin [6]. We also follow Velleman in separating out the following subsidiary definition, although we change the terminology slightly to emphasise the order preservation property.

Definition 36. *A function $\pi : \alpha \to \beta$ between two ordinals is a successor, limit, zero and order preserving (SLOOP) function if*

- *for all $\gamma < \alpha$, $\pi(\gamma + 1) = \pi(\gamma) + 1$, and*
- *for all limit $\gamma < \alpha$, $\pi(\gamma)$ is also a limit, and*
- *$\pi(0) = 0$, and*
- *for all $\gamma < \delta < \alpha$, $\pi(\gamma) < \pi(\delta)$.*

Note in particular that the composition of SLOOP functions will yield a SLOOP function.

Morasses will be defined based on a set $\mathcal{S}$ of ordered pairs of ordinals. For such an ordered pair $x = \langle \alpha, \beta \rangle$, let us denote α by $l(x)$ (the *level* of x), and β by $o(x)$ (the *order* of x). Also, note that we here use the word "tree" in the liberal sense where others might use "forest": our tree will have many root nodes.

Definition 37. *For any uncountable regular cardinal* κ*, a* $(\kappa, 1)$-morass *(or simply* morass *when* κ *is clear from the context) consists of:*

i. *a subset* $\mathcal{S}$ *of* $(\kappa \times \kappa) \cup (\{\kappa\} \times \kappa^+)$*, and*

ii. *A tree order* $\prec\!\!\!\!\!\vdash$ *on* $\mathcal{S}$*, and*

iii. *For every pair* $\langle x, y\rangle$ *of elements of* $\mathcal{S}$ *with* $x \prec\!\!\!\!\!\vdash y$*, a function* $\pi_{xy} : o(x) + 1 \to o(y) + 1$*,*

such that the following conditions hold.

Left-alignment *For each* $\alpha \leq \kappa$*, let* $\theta_\alpha = \{\beta \mid \langle \alpha, \beta\rangle \in \mathcal{S}\}$*. Then* θ_α *is in fact an ordinal. Moreover,* $\theta_\kappa = \kappa^+$*, and for* $\alpha < \kappa$*,* $0 < \theta_\alpha < \kappa$*.*

Monotonicity *For* x *and* y *in* $\mathcal{S}$*,* $x \prec\!\!\!\!\!\vdash y$ *implies* $l(x) < l(y)$*.*

Commutativity *If* $x \prec\!\!\!\!\!\vdash y \prec\!\!\!\!\!\vdash z$ *then* $\pi_{yz} \circ \pi_{xy} = \pi_{xz}$*.*

M.1 *For each pair* $x, y \in \mathcal{S}$ *with* $x \prec\!\!\!\!\!\vdash y$*,* π_{xy} *is an SLOOP function, and* $\pi_{xy}(o(x)) = o(y)$*.*

M.2 *Suppose* $x \prec\!\!\!\!\!\vdash y \in \mathcal{S}$ *and* $\nu < o(x)$*. Let* $w = \langle l(x), \nu\rangle$ *and* $z = \langle l(y), \pi_{xy}(\nu)\rangle$*. Then* $w \prec\!\!\!\!\!\vdash z$ *and* $\pi_{wz} = \pi_{xy} \restriction (\nu + 1)$*.*

M.3 *For all* $y \in \mathcal{S}$*,* $\{l(x) \mid x \prec\!\!\!\!\!\vdash y\}$ *is closed in* $l(y)$*.*

M.4 *For all* $y \in \mathcal{S}$*, if* $o(y) + 1 \neq \theta_{l(y)}$ *(as defined for* Left-alignment *above), then* $\{l(x) \mid x \prec\!\!\!\!\!\vdash y\}$ *is unbounded in* $l(y)$*. In particular, if* α *is a successor ordinal, then* $\theta_\alpha = 1$*.*

M.5 *For all* $y \in \mathcal{S}$*, if* $\{l(x) \mid x \prec\!\!\!\!\!\vdash y\}$ *is unbounded in* $l(y)$*, then* $o(y) = \bigcup\{\pi_{xy}``o(x) \mid x \prec\!\!\!\!\!\vdash y\}$*.*

M.6 *Suppose* $x \prec\!\!\!\!\!\vdash y \in \mathcal{S}$ *and* $o(x)$ *is a limit ordinal. Let* $\nu = \sup(\pi_{xy}``o(x))$ *and let* $z = \langle l(y), \nu\rangle$*. Then* $x \prec\!\!\!\!\!\vdash z$ *and* $\pi_{xz} \restriction o(x) = \pi_{xy} \restriction o(x)$*.*

M.7 *Suppose* $x \prec\!\!\!\!\!\vdash y \in \mathcal{S}$*,* $l(x) < \alpha < l(y)$*,* $o(x)$ *is a limit ordinal, and* $o(y) = \sup(\pi_{xy}``o(x))$*. If*

$$\forall \nu < o(x)\, \exists \gamma\, (\langle \alpha, \gamma\rangle \prec\!\!\!\!\!\vdash \langle l(y), \pi_{xy}(\nu)\rangle),$$

then there is a γ *such that* $\langle \alpha, \gamma\rangle \prec\!\!\!\!\!\vdash y$*.*

Note in particular axiom M.5, in the case where $l(y) = \kappa$: for any $\tau < \kappa^+$, we have τ expressed as the (increasing, by commutativity) union of the images of the maps $\pi_{x\langle\kappa,\tau\rangle} \upharpoonright o(x)$ for $x \ltimes \langle\kappa, \tau\rangle$. Further note that this isn't just a variant of κ many things "adding up" to κ, and then τ being bijective with κ, but something more direct: the maps $\pi_{x\langle\kappa,\tau\rangle}$ are order preserving, and so the ordinals mapping into τ must to some extent reflect the structure of τ. For example, if τ is a successor ordinal and so has a largest element, then for $x \ltimes \langle\kappa, \tau\rangle$ with $l(x)$ sufficiently large, $o(x)$ must also have a largest element. Also observe that for any $x \ltimes y$ in $\mathcal{S}$, $o(x) \leq o(y)$ since by M.1, π_{xy} is a strictly order-preserving function from $o(x) + 1$ to $o(y) + 1$.

When we force morasses to exist in the presence of multiple large cardinals, it will be convenient for the sake of preservation of the large cardinal property to use a partial order which lends itself to homogeneity arguments (see Section 3.4 below). The upshot will be that the generic morass will satisfy a useful property not possessed by the morasses obtained by forcing with Velleman's partial order defined in [16]. We give here a name for the kind of morass that we obtain.

Definition 38. *Suppose $M = \langle\mathcal{S}, \ltimes, \langle\pi_{xy}\rangle_{x\ltimes y}\rangle$ is a $(\kappa, 1)$-morass. An ordinal $\alpha < \kappa$ is a* mangal *of M if for all $x, y \in \mathcal{S}$ with $x \ltimes y$ and $l(x) < \alpha < l(y)$, there is a $z \in \mathcal{S}$ with $l(z) = \alpha$ such that $x \ltimes z \ltimes y$. If the set of mangals of M is cofinal in κ, we say that M is a κ-*mangrove*, or simply* mangrove *when κ is clear from the context.*

An immediate question is whether the existence of a morass implies the existence of a mangrove. It turns out that the answer is yes, as we shall show in Section 3.3, using Velleman's theorem that the existence of a morass is equivalent to a certain forcing axiom.

Another natural question is what the set of mangals of a morass can look like. A first result in this direction is the following.

Proposition 39. *For any morass M, the set of mangals of M is closed in κ.*

Proof. Suppose that $\gamma < \kappa$ is a limit point of the set of mangals of a morass $M = \langle\mathcal{S}, \ltimes, \pi\rangle$, and let $x, y \in \mathcal{S}$ satisfy $x \ltimes y$ and $l(x) < \gamma < l(y)$. For each mangal α of M such that $l(x) < \alpha < \gamma$, there is a z_α with $l(z_\alpha) = \alpha$ and $x \ltimes z_\alpha \ltimes y$. But then by M.3, there must be some z_γ with $l(z_\gamma) = \gamma$ and $x \ltimes z_\gamma \ltimes y$; thus, γ is also a mangal of M. $\dashv$

In particular, if M is a mangrove, the set of mangals of M is a closed unbounded subset of κ.

3.2 Forcing them to exist

As is the case for many combinatorial structures, one can force with a partial order of partial morasses to get a morass in the generic extension, as we shall show below. We use a partial order similar to those described in [8] and [16]. However, ours will differ in that we strengthen the requirements for an extension of a condition, ensuring that each condition of the generic goes up to a mangal of the ultimate morass. Naturally, this will make our generic morass a mangrove, whereas it is not hard to check that a generic morass for Velleman's partial order in [16] will be far from a mangrove; indeed, it will be κ-branching at the node $\langle 0, 0\rangle$, and so cannot have any mangals at all! However, many of the details of our proof will remain essentially the same as in that paper. In particular, we repeatedly use the basic construction given there for extending morass conditions, tweaking it to fit the particular requirements in each case.

3.2.1 Definitions

Let $x \ltimes_i y$ denote that y is an immediate $\ltimes$-successor of x.

Definition 40. *For any uncountable regular cardinal κ, a $(\kappa, 1)$-*morass condition *consists of:*

i. a subset $\mathcal{S}$ of $((\lambda+1)\times\kappa)\cup(\{\kappa\}\times\kappa^+)$ for some $\lambda<\kappa$, and

ii. A tree order $\ltimes$ on $\mathcal{S}$, and

iii. For every pair $\langle x, y\rangle$ of elements of $\mathcal{S}$ with $x \ltimes y$, a function $\pi_{xy} : o(x)+1 \to o(y)+1$,

such that

1. Left-alignment *holds for all $\alpha \leq \lambda$, and the set $S = \{\beta \mid \langle\kappa, \beta\rangle \in \mathcal{S}\}$ contains* 0 *and is closed under ordinal successors and predecessors.*
2. *Let f be the order-preserving bijection from $ot(S)$ to S. Then $ot(S) \leq \theta_\lambda$, and for each $\nu < ot(S)$, $\langle\lambda, \nu\rangle \ltimes \langle\kappa, f(\nu)\rangle$.*
3. Monotonicity*,* Commutativity*,* M.1*,* M.2*,* M.3 *and* M.5 *hold. Axiom* M.4 *holds for those $y \in \mathcal{S}$ such that $l(y) \leq \lambda$. Axioms* M.6 *and* M.7 *hold for those x and y such that $x \ltimes_i y$ and $l(y) \leq \lambda$.*

For our analysis, we will need to extend the new notion of a mangal to also be applicable to morass conditions.

Definition 41. *Let* $p = \langle \mathcal{S}, \mathrel{\vdash\!\!<}, \pi \rangle\rangle$ *be a morass condition. An ordinal* $\alpha \leq \lambda$ *is a* mangal *of* p *if for all* $x, y \in \mathcal{S}$ *with* $x \mathrel{\vdash\!\!<} y$ *and* $l(x) < \alpha < l(y)$*, there is a* $z \in \mathcal{S}$ *with* $l(z) = \alpha$ *such that* $x \mathrel{\vdash\!\!<} z \mathrel{\vdash\!\!<} y$.

Our definition of a morass condition is identical to that used in [16]; our partial order will differ in the definition of $\leq$. For this reason, we continue to refer to these conditions as *morass* conditions, even though we shall call our partial order the *mangrove* forcing.

Since it comes up very often, and the meaning is fairly clear, we shall consistently abuse notation, writing $x \in p$ to mean that x is an element of the set $\mathcal{S}$ for p.

A few comments about the definition of morass conditions are in order at this point. Note that requiring Axiom M.2 when $l(y) = \kappa$ implicitly entails imposing the condition on S and the various π_{xy}, that for any $x \mathrel{\vdash\!\!<} y$ with $l(y) = \kappa$, $\pi_{xy}``(o(x) + 1) \subset S$. Requiring M.7 for $x \mathrel{\vdash\!\!<}_i y$ is the same as positing the non-existence of α fitting the antecedent of that axiom, since $x \mathrel{\vdash\!\!<} \langle \alpha, \gamma \rangle \mathrel{\vdash\!\!<} y$ contradicts $x \mathrel{\vdash\!\!<}_i y$.

Observe that by requirement 1 above, $\mathrm{ot}(S)$ must be a limit ordinal; hence by requirement 2, $\theta_\lambda > 1$, and so by M.4, λ is a limit ordinal. Also, from M.2, requirement 2, and the fact that $\mathrel{\vdash\!\!<}$ is a tree order, we get that for $x \mathrel{\vdash\!\!<} y$ with $l(x) = \lambda$ and $l(y) = \kappa$, $\pi_{xy} = f \upharpoonright o(x) + 1$. Similarly, because $\mathrel{\vdash\!\!<}$ is required to be a tree order and f is surjective, one can only have $x \mathrel{\vdash\!\!<} y$ with $l(y) = \kappa$ when either $l(x) = \lambda$, or there is a z such that $x \mathrel{\vdash\!\!<} z \mathrel{\vdash\!\!<} y$ with $l(z) = \lambda$. Thus, λ is a mangal of the morass condition; allowing this trivial case in the definition of a mangal will be convenient for later arguments. As a result of λ being a mangal, it is equivalent to only explicitly require M.3 to hold for those $y \in \mathcal{S}$ with $l(y) < \kappa$. The same is trivially true of M.5, which for morass conditions is vacuous in the $l(y) = \kappa$ case. Finally, note that because of the closure properties of S given in requirement 1, f and f^{-1} are SLOOP functions.

For any regular cardinal κ, let P_κ be the set of $(\kappa, 1)$-morass conditions, along with an extra point $\mathbf{1}$ to act as a maximum element; we will generally ignore $\mathbf{1}$, it being trivial to extend the definitions to encompass it as a special case. For each non-$\mathbf{1}$ element p of P_κ, let us denote with superscript p the components and defined notions of p: $\mathcal{S}^p$, $\mathrel{\vdash\!\!<}^p$, π^p_{xy} for $x \mathrel{\vdash\!\!<}^p y$, λ^p, S^p as in requirement 1, f^p as in requirement 2, and θ^p_α. We shall also refer to $\{x \in p \mid l(x) = \lambda^p\}$ as the *top level* of the condition p.

Definition 42. *For* p *and* q *in* P*, we say that* $q \leq p$ *if*

$$\mathcal{S}^p \subseteq \mathcal{S}^q,$$

$$\mathcal{S}^p \cap ((\lambda^p + 1) \times \kappa) = \mathcal{S}^q \cap ((\lambda^p + 1) \times \kappa),$$

$$\mathrel{\vdash\!\!\!<}^p = \mathrel{\vdash\!\!\!<}^q \restriction \mathcal{S}^p,$$

$$\forall x \mathrel{\vdash\!\!\!<}^p y \in \mathcal{S}^p(\pi^p_{xy} = \pi^q_{xy}),$$

and λ^p *is a mangal of* q.

Notice in particular that a condition p may be extended to a condition q simply by extending S^p and f^p — indeed, this is the only way to extend p without changing λ. This kind of extension is analogous to the possibility in Velleman's partial order of extending p by increasing θ_{λ^p}, but we avoid the necessity of (and indeed rule out) adding edges to the tree below level λ^p. We shall sometimes refer to the last requirement above (which is what is new in this forcing) as simply the *mangal requirement*; requiring λ^p to be a mangal of q for $q \leq p$ will ensure that λ^p is a mangal of the generic morass if p is in the generic (see below).

The only slight difficulty in seeing that $\leq$ as defined above is transitive is with the mangal requirement — whether λ^p is a mangal of r if $p \leq q \leq r$. This is easily resolved, however:

Lemma 43. *The relation* $\leq$ *as given in Definition 42 is transitive.*

Proof. Suppose we have three conditions p, q and r in P_κ such that $p \leq q \leq r$; we wish to show that λ^p is a mangal of r. So suppose there are x and y in r such that $x \mathrel{\vdash\!\!\!<}^r y$ and $l(x) < \lambda^p < l(y)$. If $l(y) \leq \lambda^q$, then $y \in q$ and so since λ^p is a mangal of q, there is a $z \in q$ (and hence $z \in r$ also) such that $l(z) = \lambda^p$ and $x \mathrel{\vdash\!\!\!<}^q z \mathrel{\vdash\!\!\!<}^q y$, whence $x \mathrel{\vdash\!\!\!<}^r z \mathrel{\vdash\!\!\!<}^r$, as required.

If $l(y) > \lambda^q$, then because λ^q is a mangal of r, there is some $z' \in r$ such that $l(z') = \lambda^q$ and $x \mathrel{\vdash\!\!\!<}^r z' \mathrel{\vdash\!\!\!<}^r y$. But now we are reduced to the previous situation with z' replacing y, and so there is a $z \in r$ with $l(z) = \lambda^p$ and $x \mathrel{\vdash\!\!\!<}^r z \mathrel{\vdash\!\!\!<}^r z'$. Transitivity of $\mathrel{\vdash\!\!\!<}^r$ then gives $x \mathrel{\vdash\!\!\!<}^r z \mathrel{\vdash\!\!\!<}^r y$. ⊣

Therefore, letting $\mathbb{P}_\kappa = \langle P_\kappa, \leq \rangle$, we have that $\mathbb{P}_\kappa$ is a partial order — the *Mangrove Partial Order* or *Mangrove Forcing* at κ. In this section (3.2) we will generally omit the subscript κ, as κ will be unvarying. Moreover, when we say that morass conditions are compatible or incompatible, or comparable or incomparable, we will mean with respect to this mangrove partial order, rather than Velleman's morass partial order. We will also abuse notation, writing $p \in \mathbb{P}$ for $p \in P_\kappa$. The goal of the section, realised in Theorem 55, is to show that forcing with $\mathbb{P}$ yields a mangrove in the generic extension.

3.2.2 A first "bamboo construction"

It is not yet clear that there are any elements of $\mathbb{P}$ other than **1**. The following lemma constructs one, and the basic technique of the construction, modified from one given in [16], will be reused multiple times as we wish to extend conditions in various ways later on. We call such a construction a "bamboo construction" because of the straight, vertical, unbranching branches that it produces.

Lemma 44. *There is a morass condition q with $S^q = \omega$.*

Proof. We start by defining θ^q_α for α up to ω^ω (where the exponentiation is as ordinals), which will be our λ^q. Let

$$\theta_\alpha = \begin{cases} \{n \mid \exists\zeta(\alpha = \omega^n \cdot \zeta)\} & \text{if } \alpha < \omega^\omega \\ \omega & \text{if } \alpha = \omega^\omega \end{cases}$$

Now let

$$\mathcal{S} = (\{\kappa\} \times \omega) \cup \bigcup_{\alpha \leq \omega^\omega} (\{\alpha\} \times \theta_\alpha),$$

$$x \ltimes^q y \longleftrightarrow l(x) < l(y) \wedge o(x) = o(y),$$

and for $x \ltimes^q y$ and all $\gamma \leq o(x)$,

$$\pi^q_{xy}(\gamma) = \gamma.$$

We claim that q so defined is a morass condition. The set $\mathcal{S}$ clearly has the right form, $\ltimes^q$ is a (non-branching!) tree order, and for $x \ltimes^q y$, π^q_{xy} is a function with the right domain and range. Requirements 1 and 2 for a morass condition are immediate from the construction, as are Monotonicity, Commutativity, M.1, M.2, M.5, M.6 and M.7, the latter two being vacuous in the present context. Finally, axioms M.3 and M.4 below level κ reduce to simple properties of ordinal arithmetic: respectively, the limit of ordinals divisible by ω^n is divisible by ω^n; and if γ is divisible by ω^m for $m > n$, then the set of ordinals in γ divisible by ω^n is unbounded in γ. ⊣

The reader should note that such a bamboo construction does not reflect the frequent branching that must occur in a mangrove to get κ^+-many leaves at level κ. However, while one might not expect to find bamboo in a mangrove, it does make a convenient building material.

3.2.3 μ-equivalence

To acquaint the reader better with morass conditions and mangrove forcing, we now present some lemmas that may help with intuition and which will be useful later in the chapter. First, we define a "sameness" notion which appears frequently, and which will be crucial for the homogeneity arguments of Section 3.4.

Definition 45. *Two morass conditions p and q are μ-*equivalent*, $p \overset{\mu}{\sim} q$, if they are "the same up to level μ", that is,*

$$\begin{aligned} \mathcal{S}^p \cap ((\mu+1)\times\kappa) &= \mathcal{S}^q \cap ((\mu+1)\times\kappa), \\ \mathrel{|\!\prec}^p \cap((\mu+1)\times\kappa)^2 &= \mathrel{|\!\prec}^q \cap((\mu+1)\times\kappa)^2, \end{aligned}$$

for $x \mathrel{|\!\prec}^p y$ with $l(y) \leq \mu$,

$$\pi^p_{xy} = \pi^q_{xy},$$

and μ is a mangal of both p and q.

Note that we include level μ in "up to level μ." Once again we shall refer to the last requirement in this definition as the mangal requirement.

Clearly $\overset{\mu}{\sim}$ is an equivalence relation on the set of those conditions with $\lambda \geq \mu$ and μ as a mangal (although it is not even reflexive for other conditions). While the presence of the mangal requirement means that $\mu \leq \nu$ does not yield $p \overset{\nu}{\sim} q \to p \overset{\mu}{\sim} q$, we do have the following.

Lemma 46. *If $\mu \leq \nu$ and $p \overset{\mu}{\sim} q$ and $q \overset{\nu}{\sim} r$ for some elements p, q, r of $\mathbb{P}$, then $p \overset{\mu}{\sim} r$.*

Proof. This is essentially the same as Lemma 43, showing that the $\leq$ relation of $\mathbb{P}$ is transitive. Again, the only difficulty is in showing that μ is a mangal of r. Suppose $x, y \in r$ are such that $x \mathrel{|\!\prec}^r y$ and $l(x) < \mu < l(y)$. If $l(y) \leq \nu$, then $y \in q$ with $x \mathrel{|\!\prec}^q y$. Thus, since μ is a mangal of q, there is a $z \in q$ with $l(z) = \mu$ and $x \mathrel{|\!\prec}^q z \mathrel{|\!\prec}^q y$. So $z \in r$ and $x \mathrel{|\!\prec}^r z \mathrel{|\!\prec}^r y$.

If $l(y) > \nu$, then since ν is a mangal of r, there is a $z \in r$ with $l(z) = \nu$ and $x \mathrel{|\!\prec}^r z \mathrel{|\!\prec}^r y$. By transitivity of $\mathrel{|\!\prec}^r$, we are reduced to the case above. ⊣

Also, it is immediate from the definitions that $q \leq p \to q \overset{\lambda^p}{\sim} p$. A stronger statement is in fact true.

Lemma 47. *Let $p, q \in \mathbb{P}$ be (mangrove-) compatible morass conditions, with $\lambda^p \leq \lambda^q$. Then $p \overset{\lambda^p}{\sim} q$.*

Proof. Let $r \in \mathbb{P}$ be such that $r \leq p$ and $r \leq q$. It is immediate from the fact that $p \overset{\lambda^p}{\sim} r$ and $q \overset{\lambda^q}{\sim} r$ that the necessary equalities for $p \overset{\lambda^p}{\sim} q$ hold. To see that λ^p is a mangal of q, observe that since λ^p is a mangal of r, we have that for any x and y in q with $l(x) < \lambda^p < l(y)$,

$$\begin{aligned} x \mathrel{|\!\prec}^q y &\longleftrightarrow x \mathrel{|\!\prec}^r y \\ &\longleftrightarrow \exists z \in r(l(z) = \lambda^p \wedge x \mathrel{|\!\prec}^r z \mathrel{|\!\prec}^r y) \\ &\longleftrightarrow \exists z \in q(l(z) = \lambda^p \wedge x \mathrel{|\!\prec}^q z \mathrel{|\!\prec}^q y), \end{aligned}$$

so λ^p is also a mangal of q. ⊣

Note that, in the situation of the lemma, the only possible obstruction to $q \leq p$ holding is that f^p might not factor through f^q. Indeed, S^p need not be a subset of S^q, in which case f^p certainly cannot factor through f^q.

One reason that the notion of μ-equivalent is important, particularly in the case where $\mu = \lambda^p$ for some condition p, is that most of the "morass-like" requirements for the condition p only pertain to the part of the condition below λ^p. The following lemma makes this observation concrete.

Lemma 48. *Let p be a morass condition, and let S' be a subset of κ^+ containing 0, closed under ordinal successors and predecessors, and such that $ot(S') \leq \theta^p_{\lambda^p}$. Then there is a unique morass condition q such that $\lambda^q = \lambda^p$, $q \overset{\lambda^p}{\sim} p$, and $S^q = S'$.*

Proof. The description of q in the statement of the lemma completely determines q: to get from p to q we simply change S^p to S', correspondingly replace f^p with the order-preserving bijection $f^q : \mathrm{ot}(S') \to S'$, and modify $\ltimes$ and π as appropriate for the new f. To be precise,

$$\mathcal{S}^q = \big(\mathcal{S}^p \cap ((\lambda^p + 1) \times \kappa)\big) \cup (\{\kappa\} \times S'),$$

$$\begin{aligned} x \ltimes^q y \quad \longleftrightarrow \quad & \big(l(x) < l(y) \leq \lambda^p \ \wedge \ x \ltimes^p y\big) \vee \\ & \big(l(x) = \lambda^p \wedge l(y) = \kappa \ \wedge \ f^q(o(x)) = o(y)\big) \vee \\ & \big(l(x) < \lambda^p \wedge l(y) = \kappa \ \wedge \ \exists z \in \mathcal{S}^q (l(z) = \lambda^p \wedge x \ltimes^q z \ltimes^q y)\big), \end{aligned}$$

and for $x, y \in \mathcal{S}^q$ such that $x \ltimes^q y$,

$$\pi^q_{xy} = \begin{cases} \pi^p_{xy} & \text{if } l(x) < l(y) \leq \lambda^p \\ f \restriction (o(x) + 1) & \text{if } l(x) = \lambda^p \text{ and } l(y) = \kappa \\ \pi^q_{zy} \circ \pi^q_{xz} & \text{if } l(x) < \lambda^p, l(y) = \kappa, l(z) = \lambda^p \text{ and } x \ltimes^q z \ltimes^q y. \end{cases}$$

Every aspect of this definition is clearly necessary for q to be a morass condition of the desired form, giving the claimed uniqueness. If we can further show that this definition is sufficient to make q a morass condition we will be done. But now the restricted forms of M.3–M.7 that are required all hold because they do in p (recall from the discussion after Definition 40 that M.3 and M.5 need only be checked up to level λ). The rest of the requirements for a morass condition are almost all immediate from the definitions; the only points worth mentioning are that f^q is a SLOOP function, and that M.2 in the case that $l(x) < \lambda^p < l(y) = \kappa$ follows from Commutativity and and M.2 in the other two cases of the definition of π^q_{xy}. $\dashv$

In general this q and the original p will not be compatible — if there is a $\tau < \kappa^+$ and $\alpha \neq \beta < \theta^p_{\lambda^p}$ such that $f^p(\alpha) = \tau = f^q(\beta)$, there can be no common extension r, as it would need to satisfy both $\langle \lambda^p, \alpha\rangle \ltimes^r \langle \kappa, \tau\rangle$ and $\langle \lambda^p, \beta\rangle \ltimes^r \langle \kappa, \tau\rangle$, violating the fact that $\ltimes^r$ is a tree relation and Monotonicity holds in r. On the other hand, in Lemma 51 we shall see a circumstance in which such τ, α and β do not exist, and p and q can shown to be compatible.

3.2.4 Cardinal preservation

Our claim is that forcing with $\mathbb{P}$ will yield a mangrove with height κ, with of course $\theta_\kappa = \kappa^+$. For this it is important that the cardinals κ and κ^+ are preserved. In fact, if we assume that we have the GCH in our original model, all cardinals will be preserved, thanks to closure and chain condition properties of $\mathbb{P}$ which we describe here.

Recall that a partial order P is *κ-closed* if every descending chain of conditions from P of length less than κ has a lower bound in P. Further recall that if P is κ-closed, then forcing with $\mathbb{P}$ adds no new sequences of length less than κ, and so in particular all cardinals $\leq \kappa$ are preserved. In our particular circumstance, knowing that $\mathbb{P}$ is κ-closed will also be useful for "gluing together" successive extensions, making it possible to show that a variety of subsets of $\mathbb{P}$ are dense using just a few basic extension lemmas repeatedly.

One may show that $\mathbb{P}$ is κ-closed by the natural argument, taking the union of the given chain of conditions, and adding a top level as appropriate for the new κ-th level. As pointed out in [16] though, the same argument actually yields a stronger property for $\mathbb{P}$, one which will be useful when showing later that morasses give rise to mangroves. Perhaps more significantly, it will be exactly what we require in order to obtain master conditions when we force to obtain mangroves while preserving large cardinals.

We make a preliminary definition in order to give Proposition 50 the necessary strength for the argument of Section 3.3.

Definition 49. *For any $\alpha \leq \kappa^+$, $\mathbb{P}_{\kappa,\alpha}$ denotes the suborder of $\mathbb{P}$ consisting of those conditions $p \in \mathbb{P}$ such that $S^p \subseteq \alpha$.*

One might wonder whether the inclusion $\mathbb{P}_{\kappa,\alpha} \hookrightarrow \mathbb{P}$ is a complete embedding, as defined for example in [14], page 218. It turns out that this depends on α — see Proposition 62 in Section 3.3.1.

Recall that a set Y in a partial order is *directed* if for every p and q in Y, there is an r in Y such that $r \leq p$ and $r \leq q$. Recall further that a partial order P is *κ-directed-closed* if every directed subset Y of P with cardinality less than κ has a lower bound in P. Note in particular that the

κ in "κ-directed-closed" refers to the level of closure rather than the level of directedness. Of course, κ-directed-closure implies κ-closure.

Proposition 50. *For every ordinal $\alpha \leq \kappa^+$, the poset $\mathbb{P}_{\kappa,\alpha}$ is κ-directed-closed. In particular, $\mathbb{P} = \mathbb{P}_{\kappa,\kappa^+}$ is κ-closed, and forcing with $\mathbb{P}$ adds no new sequences of length less than κ of elements of V.*

Proof. Suppose that Y is a directed set in $\mathbb{P}_{\kappa,\alpha}$ with $|Y| = \gamma$ for some cardinal $\gamma < \kappa$. Let $\bar{q}$ be the direct limit of these structures — that is, let $\mathcal{S}^{\bar{q}} = \bigcup_{p\in Y} \mathcal{S}^p$, $x \mathrel{\vdash\!<}^{\bar{q}} y$ if and only if for some $p \in Y$ containing both x and y, $x \mathrel{\vdash\!<}^p y$, and in this case $\pi^{\bar{q}}_{xy} = \pi^p_{xy}$. Note that $\bar{q}$ is well defined: if $p, r \in Y$ both contain x and y, then a common extension $s \in Y$ will also contain x and y, and by the definition of $\leq$ in $\mathbb{P}$, p and r must both agree with s regarding whether $x \mathrel{\vdash\!<} y$, and on the function π_{xy} when $x \mathrel{\vdash\!<} y$ does hold.

Suppose that $\bar{q}$ has a non-empty top level $\lambda^{\bar{q}}$. Then there exists at least one condition p in Y such that $\lambda^p = \lambda^{\bar{q}}$. Recall that the only possibility for extending a condition p without altering λ is to end extend S^p and f^p. Since Y is directed, it follows that conditions p and r in Y with $\lambda^p = \lambda^r = \lambda^{\bar{q}}$ must be comparable: an $s \in Y$ extending both simply has an S^s end-extended from each of S^p and S^r, and so the larger of the latter two sets must simply end-extend the smaller. Thus, the subset $Y_{\lambda^{\bar{q}}}$ of Y consisting of all elements p of Y with $\lambda^p = \lambda^{\bar{q}}$ forms a decreasing chain. Moreover, every element of Y is extended by an element of $Y_{\lambda^{\bar{q}}}$, so $\bar{q}$ is the direct limit of the chain $Y_{\lambda^{\bar{q}}}$. In this case it is straightforward to verify that all of the requirements for a morass condition hold for $\bar{q}$, simply because they do for each $p \in Y_{\lambda^{\bar{q}}}$. Similarly, it is immediate from the definitions that $\bar{q}$ is an extension of each $p \in Y_{\lambda^{\bar{q}}}$, and hence of each $p \in Y$. Finally, since we have simply taken a union to obtain $S^{\bar{q}}$, $S^{\bar{q}} \subseteq \alpha$ and $\bar{q}$ is a lower bound for Y in $\mathbb{P}_{\kappa,\alpha}$.

So suppose now that $\bar{q}$ does not have a top level; then it is not a morass condition and we must modify it to obtain one. Letting $\lambda^q = \sup_{p\in Y}(\lambda^p)$, $S^q = \bigcup_{p\in Y} S^p = \{x \in \mathcal{S}^{\bar{q}} \mid l(x) = \kappa\}$, and f^q be the order preserving bijection from $\operatorname{ot}(S^q)$ to S^q, define q by

$$\mathcal{S}^q = \mathcal{S}^{\bar{q}} \cup (\{\lambda^q\} \times \operatorname{ot}(S^q)),$$

$$\begin{aligned}
x \mathrel{\vdash\!<}^q y \quad \longleftrightarrow \quad & \Big(x \mathrel{\vdash\!<}^{\bar{q}} y\Big) \vee \\
& \Big(l(x) = \lambda^q \wedge l(y) = \kappa \wedge f^q(o(x)) = o(y)\Big) \vee \\
& \Big(l(x) < \lambda^q \wedge l(y) = \lambda^q \wedge \\
& \qquad \exists z\big(l(z) = \kappa \wedge x \mathrel{\vdash\!<}^{\bar{q}} z \wedge f^q(o(y)) = o(z)\big)\Big),
\end{aligned}$$

and for $x \mathrel{\vdash\!\!<}^q y$,

$$\pi^q_{xy} = \begin{cases} \pi^{\bar{q}}_{xy} & \text{if } x \mathrel{\vdash\!\!<}^{\bar{q}} y, \\ f^q \restriction (o(x)+1) & \text{if } l(x) = \lambda^q \text{ and } l(y) = \kappa, \\ (f^q)^{-1} \circ \pi^{\bar{q}}_{xz} & \text{if } l(y) = \lambda^q,\ l(z) = \kappa \text{ and } x \mathrel{\vdash\!\!<}^{\bar{q}} z. \end{cases}$$

We claim that with this definition q is a morass condition. It is clear that $\mathcal{S}^q$ has the right form, $\mathrel{\vdash\!\!<}^q$ is a tree order, and each π^q_{xy} has the right domain and range. Requirement 2 for a morass condition is also immediate from the definition of q. Requirement 1 follows from the fact that it is true for each $p \in Y$ and the easy observation that it also holds at the new top level. Monotonicity, Commutativity and M.1 are similarly straightforward to verify.

For M.2, the cases where both $l(x)$ and $l(y)$ are less than λ^q or where $l(x) < \lambda^q$ and $l(y) = \kappa$ follow from M.2 for the conditions $p \in Y$. If $l(x) = \lambda^q$ and $l(y) = \kappa$, the fact that M.2 is satisfied is immediate from the definition of q. Finally, the case of $l(x) < \lambda^q$ and $l(y) = \lambda^q$ is straightforward from the fact that M.2 is satisfied in some $p \in Y$ containing x and z, where $x \mathrel{\vdash\!\!<}^q y \mathrel{\vdash\!\!<}^q z$: if $\nu < o(x)$, then

$$\langle l(x), \nu \rangle \mathrel{\vdash\!\!<}^p \langle \kappa, \pi^p_{xz}(\nu) \rangle$$

and so

$$\langle l(x), \nu \rangle \mathrel{\vdash\!\!<}^q \langle \lambda^q, (f^q)^{-1} \circ \pi^p_{xz}(\nu) \rangle = \langle l(y), \pi^q_{xy}(\nu) \rangle$$

with

$$\pi^q_{\langle l(x),\nu\rangle \langle l(y), \pi^q_{xy}(\nu)\rangle} = (f^q)^{-1} \circ \pi^p_{xz} \restriction (\nu+1) = \pi^q_{xy} \restriction (\nu+1)$$

as required.

In the verifications of both M.3 and M.4, the only case we need to consider is when $l(y) = \lambda^q$, since for any $\alpha < \lambda^q$, any condition $p \in Y$ with $\lambda^p \geq \alpha$ fixes the structure of the tree below level α. So suppose that $l(y) = \lambda^q$, and that $y \mathrel{\vdash\!\!<}^q z$ with $l(z) = \kappa$. Then there is some $p_z \in Y$ such that $z \in p_z$. For every $r \in Y$ extending p_z,

$$\begin{aligned} \{l(x) \mid x \mathrel{\vdash\!\!<}^q y\} \cap (\lambda^r + 1) &= \{l(x) \mid x \mathrel{\vdash\!\!<}^q z\} \cap (\lambda^r + 1) \\ &= \{l(x) \mid x \mathrel{\vdash\!\!<}^r z\} \end{aligned}$$

which is closed in $\lambda^r + 1$ by M.3 for r. Moreover, $\lambda^r \in \{l(x) \mid x \mathrel{\vdash\!\!<}^r z\}$ for each $r \leq p_z$ in Y since $l(z) = \kappa$. Because $\lambda^q = \sup_{p \in Y}(\lambda^p)$ and Y is directed, we have that the set $\{l(x) \mid x \mathrel{\vdash\!\!<}^q y\}$ is closed and unbounded in λ^q, as desired for M.3 and M.4. Also, this implies that for y with $l(y) = \lambda^q$, it is never the case that $x \ltimes_i^q y$. Hence, the restricted forms of M.6 and M.7 that we require are vacuous at level λ^q, and of course hold below that level because they do in each p_α.

It therefore only remains to check M.5, and again, it is sufficient to consider the case when $l(y) = \lambda^q$. Suppose $\mu < o(y)$; we wish to show that there is an $x \ltimes^q y$ and a $\nu < o(x)$ such that $\pi^q_{xy}(\nu) = \mu$. Let $w = \langle \kappa, f^q(\mu)\rangle$, let $z = \langle \kappa, f^q(o(y))\rangle$ so that $y \ltimes^q z$, and let $p \in Y$ be such that w and z are both in p. Let $v, x \in p$ with $l(v) = l(x) = \lambda^p$ be such that $x \ltimes^p z$ and $v \ltimes^p w$. Then $o(v) < o(x)$, and

$$\pi^p_{xz}(o(v)) = f^p(o(v)) = \pi^p_{vw}(o(v)) = o(w) = f^q(\mu).$$

Now by the definition of $\ltimes^q$ we have that $x \ltimes^q y$ and $\pi^q_{xy}(o(v)) = \mu$ as required.

Overall then, we have that q is a morass condition. Moreover, we claim that $q \leq p$ for every $p \in Y$. Indeed, the only part of this claim that is not immediate from the definition of q is the mangal requirement, and even this is trivial to verify when $l(y) \neq \lambda^q$. So suppose $p \in Y$, and $x, y \in q$ with $x \ltimes^q y$, $l(y) = \lambda^q$ and $l(x) < \lambda^p$. Let $w = \langle \kappa, f^q(o(y))\rangle$; then $x \ltimes^q w$. Let $r \in Y$, $r \leq p$ be such that $w \in r$, and hence $x \ltimes^r w$. By the definition of extension, λ^p is a mangal of r, and so there is a $z \in r$ with $l(z) = \lambda^p$ such that $x \ltimes^r z \ltimes^r w$, and hence $x \ltimes^q z \ltimes^q y \ltimes^q w$. Thus, λ^p is a mangal of q for each $p \in Y$, and so $q \leq p$ for each $p \in Y$.

Hence, we have constructed a lower bound q for Y. As before, S^q is simply the union of $\{S^p \,|\, p \in Y\}$, so $S^q \subseteq \alpha$ and $q \in \mathbb{P}_{\kappa,\alpha}$. Thus, $\mathbb{P}_{\kappa,\alpha}$ is indeed κ-directed-closed. $\dashv$

Our proof that $\mathbb{P}$ satisfies the κ^+-cc will require a more complicated construction, building a lower bound for conditions which are the same below level κ but have different sets S at level κ.

Lemma 51. *Let p and q be morass conditions such that $\lambda^p = \lambda^q$ and $p \overset{\lambda^p}{\sim} q$. Suppose further that there is some $S^0 \subset \kappa^+$ such that both S^p and S^q end-extend S^0, and $\min(S^q \smallsetminus S^0) \geq \sup(S^p)$. Then p and q are compatible in $\mathbb{P}$.*

Proof. We shall build a common extension r of p and q, using a construction similar to that in the proof of Lemma 44.

Let $S^r = S^p \cup S^q$ and let f^r be the order preserving bijection from $\mathrm{ot}(S^r)$ to S^r. Let $\gamma^0 = \mathrm{ot}(S^0)$, $\gamma^p = \mathrm{ot}(S^p)$ and let γ_q be such that $\mathrm{ot}(S^q) = \gamma^0 + \gamma_q$; with this notation, $\mathrm{ot}(S^r) = \gamma^p + \gamma_q$. Note also that by requirements 1 and 2 for morass conditions, γ^0, γ^p and γ_q are limit ordinals, and $\gamma^p + \gamma_q < \kappa$.

Let $\lambda^r = \lambda^p + \omega^{\gamma^p+\gamma_q}$. For $\alpha \leq \lambda^r$, define θ^r_α by

$$\theta^r_\alpha = \begin{cases} \theta^p_\alpha & \text{if } \alpha \leq \lambda^p \\ \{\eta \,|\, \exists\zeta(\beta = \omega^\eta \cdot \zeta)\} & \text{if } \alpha = \lambda^p + \beta,\ 0 < \beta < \omega^{\gamma^p+\gamma_q} \\ \gamma^p + \gamma_q & \text{if } \alpha = \lambda^r. \end{cases}$$

Now, let

$$\mathcal{S}^r = (\{\kappa\} \times S^r) \cup \bigcup_{\alpha \leq \lambda^r} (\{\alpha\} \times \theta_\alpha),$$

and define $\ltimes^r$ and the π^r_{xy} (in each case defined only when $x \ltimes^r y$, even though we do not state that below) according to the following cases. Let id_α denote the identity function on α.

- If $l(x) \geq l(y)$ then $x \not\ltimes^r y$
- If $l(x) < l(y) \leq \lambda^p$, then $x \ltimes^r y \leftrightarrow x \ltimes^p y$, and $\pi^r_{xy} = \pi^p_{xy}$.
- If $\lambda^p < l(x) < l(y) < \kappa$, then $x \ltimes^r y \leftrightarrow o(x) = o(y)$, and $\pi^r_{xy} = \mathrm{id}_{o(x)+1}$.
- If $\lambda^p = l(x) < l(y) < \kappa$ then
  - if $o(y) < \gamma^p$ then $x \ltimes^r y \leftrightarrow o(x) = o(y)$ and $\pi^r_{xy} = \mathrm{id}_{o(x)+1}$, and
  - if $o(y) \geq \gamma^p$ then $x \ltimes^r y \leftrightarrow \exists\zeta(o(x) = \gamma^0 + \zeta \wedge o(y) = \gamma^p + \zeta)$ and

$$\pi^r_{xy}(\alpha) = \begin{cases} \alpha & \text{if } \alpha < \gamma^0 \\ \gamma^p + \eta & \text{if } \alpha = \gamma^0 + \eta. \end{cases}$$

- If $l(x) < \lambda^p < l(y) < \kappa$ then

$$x \ltimes^r y \leftrightarrow \exists z(l(z) = \lambda^p \wedge x \ltimes^r z \ltimes^r y),$$

  and $\pi^r_{xy} = \pi^r_{zy} \circ \pi^r_{xz}$.

- If $\lambda^r = l(x) < l(y) = \kappa$ then $x \ltimes^r y \leftrightarrow f^r(o(x)) = o(y)$ and $\pi^r_{xy} = f^r \restriction (o(x)+1)$.
- If $l(x) < \lambda^r < l(y) = \kappa$ then

$$x \ltimes^r y \leftrightarrow \exists z(l(z) = \lambda^r \wedge x \ltimes^r z \ltimes^r y),$$

  and $\pi^r_{xy} = \pi^r_{zy} \circ \pi_{xz}$.

We must verify that r so defined is indeed a morass condition. That $\mathcal{S}$ is of the right form, $\ltimes^r$ is a tree order and π^r_{xy} is defined where necessary with the right domain and range is clear. Requirements 1 and 2 for a morass condition as well as Monotonicity, Commutativity, M.5, and the restricted form of M.6 required are immediate from the construction; for M.6 note in particular that $\langle \lambda^p, \gamma^0 \rangle$ is $\ltimes^r$-less than both $\langle \lambda^p + \omega^{\gamma^p}, \gamma^0 \rangle$ and $\langle \lambda^p + \omega^{\gamma^p}, \gamma^p \rangle$. Once we have observed that Commutativity holds, M.2 also follows, as does M.1 with the earlier observation that γ^p and γ^0 are limit ordinals. Axioms M.3 and M.4 are satisfied as in Lemma 44 along with the fact that they hold in p and hence up to level λ^p in r. Finally, M.7 reduces to a property of ordinal arithmetic: if δ_0 is a limit ordinal, and α is divisible by ω^δ for all $\delta < \delta_0$, then α is divisible by ω^{δ_0}.

Therefore, r is a morass condition. Clearly $r \leq p$ and $r \leq q$; in particular note that the mangal requirement holds by construction. Thus, p and q are compatible. $\dashv$

Proposition 52. *(GCH) The poset $\mathbb{P}$ is κ^+-cc.*

Proof. Suppose for the sake of contradiction that A is an antichain in $\mathbb{P}$ of cardinality κ^+. Since $\lambda^p < \kappa$ for any $p \in \mathbb{P}$, we may assume without loss of generality that $\lambda^p = \lambda$ for all $p \in A$ and some $\lambda < \kappa$. Similarly, there are at most $\kappa^{|\lambda|} = \kappa$ possible functions $\theta : (\lambda + 1) \to \kappa$ determining the set $\mathcal{S}$ up to level λ, so by further paring down A we may assume that $\mathcal{S}^p \cap ((\lambda + 1) \times \kappa) = \mathcal{S}^q \cap ((\lambda + 1) \times \kappa)$ for all $p, q \in A$. Furthermore, this common initial segment of $\mathcal{S}$ will have a cardinality $\mu < \kappa$, so there are at most κ many binary relations on it, and we may assume that $\ltimes^p = \ltimes^q$ for $p, q \in A$. With $\ltimes$ fixed, there are at most $(\mu^\mu)^{\mu \times \mu} \leq \kappa$ different possibilities for the set of functions π_{xy} with $l(x) < l(y) \leq \lambda$. Hence, we may assume without loss of generality that the conditions in A are identical up to level λ. Note that this implies that they are completely determined by the set S at level κ: S determines f, which in turn determines the relation $x \ltimes y$ and maps π_{xy} for $l(y) = \kappa$.

It follows from the Δ-system Lemma (see, for example, [14], Theorem II.1.6) that there is a $B \subset A$ of cardinality κ^+ such that the set $\{S^p \mid p \in B\}$ forms a Δ-system. Moreover, B may be chosen in such a way that there is a $\gamma < \kappa$ and an $S^0 \subset \kappa^+$ such that for all $p, q \in B$, $\mathrm{ot}(S^p) = \mathrm{ot}(S^q) = \gamma$, S^p and S^q end-extend S^0 with $S^p \cap S^q = S^0$, and $\sup(S^p) < \min(S^q \setminus S^0)$ or $\sup(S^q) < \min(S^p \setminus S^0)$. With B so defined, any two of its elements will be compatible by Lemma 51. This contradicts the initial assumption that A was an antichain, and so we may conclude that $\mathbb{P}$ enjoys the κ^+-cc. $\dashv$

Note that we have in fact proved the slightly stronger property, that $\mathbb{P}$ is

κ^+-*Knaster*: for any $A \subset \mathbb{P}$ of cardinality κ^+, there is a $B \subset A$ of cardinality κ^+ whose elements are pairwise compatible.

3.2.5 A mangrove from a generic

We now turn to the main task of showing that the direct limit of the conditions in a $\mathbb{P}$-generic G is a mangrove in $V[G]$. Of course, for this we will need to show that various subsets of $\mathbb{P}$ are dense. The following extension lemmas, in the same vein as Lemma 44 and the argument for Lemma 51, will give us the requisite density.

Lemma 53. *For any $p \neq \mathbf{1} \in \mathbb{P}$ and any limit ordinal $\sigma < \kappa^+$, there is a $q \leq p$ such that $S^q = S^p \cup ((\sigma + \omega) \setminus \sigma)$.*

Proof. If $\sigma \in S^p$, then we may of course take $q = p$, so suppose $\sigma \notin S^p$. Let S^q be as in the statement of the lemma. If $\sigma \geq \sup(S^p)$ and $\mathrm{ot}(S^p) + \omega \leq \theta_{\lambda^p}$, we may simply extend p to q by end-extending f^p, making the bijection from $\mathrm{ot}(S^p) + \omega$ to S^q our f^q. Extending as necessary both the tree relation and the maps π_{xy} by composition, it is straightforward to check that this defines a $q \leq p$ with the desired S^q. However, the construction described below works in all cases, so it is not necessary to distinguish this particular case.

If the set $S^p \setminus \sigma$ is non-empty, let τ be its least element, and let $\bar{\tau}$ be the unique ordinal such that $\langle \lambda^p, \bar{\tau} \rangle \ltimes^p \langle \kappa, \tau \rangle$. Otherwise, let $\bar{\tau} = \mathrm{ot}(S^p)$. Clearly τ (if defined) and $\bar{\tau}$ are limit ordinals. Further, let ξ be such that $\bar{\tau} + \xi = \mathrm{ot}(S^p)$; then ξ is either 0 or a limit ordinal, and $\mathrm{ot}(S^q) = \bar{\tau} + \omega + \xi$.

Note that $\bar{\tau} + \omega + \xi < \kappa$, since $\bar{\tau} + \xi = \mathrm{ot}(S^p) \leq \theta_{\lambda^p} < \kappa$, and so letting $\delta = \omega^{\bar{\tau}+\omega+\xi}$, we also have $\delta < \kappa$. Let $\lambda^q = \lambda^p + \delta$, and let

$$\theta^q_\alpha = \begin{cases} \theta^p_\alpha & \text{if } \alpha \leq \lambda^p \\ \{\nu \mid \exists \zeta (\beta = \omega^\nu \cdot \zeta\} & \text{if } \alpha = \lambda^p + \beta,\ 0 < \beta < \delta \\ \bar{\tau} + \omega + \xi & \text{if } \alpha = \lambda^q. \end{cases}$$

Now let

$$\mathcal{S}^q = \left(\bigcup \{\{\alpha\} \times \theta^q_\alpha \mid \alpha \leq \lambda^q\}\right) \cup (\{\kappa\} \times S^q);$$

then the usual definitions for θ^q_α and S^q from $\mathcal{S}^q$ match the chosen values, and we also immediately have f^q, defined on the whole width $\theta^q_{\lambda^q}$ of the new top level λ^q.

Let us now define $\ltimes^q$ and the functions π^q_{xy}. This will be very similar to the definition for the construction in the argument for Lemma 51. As before, let id_α denote the identity function on α, and although it is not explicitly stated, in each case take π^q_{xy} to be defined only when $x \ltimes^q y$.

- If $l(x) \geq l(y)$ then $x \not\mathrel{|\!\!\prec}^q y$
- If $l(x) < l(y) \leq \lambda^p$, then $x \mathrel{|\!\!\prec}^q y \leftrightarrow x \mathrel{|\!\!\prec}^p y$, and $\pi^q_{xy} = \pi^p_{xy}$.
- If $\lambda^p < l(x) < l(y) < \kappa$, then $x \mathrel{|\!\!\prec}^q y \leftrightarrow o(x) = o(y)$, and $\pi^q_{xy} = \mathrm{id}_{o(x)+1}$.
- If $\lambda^p = l(x) < l(y) < \kappa$ then
  - if $o(y) \leq \bar{\tau}$ then $x \mathrel{|\!\!\prec}^q y \leftrightarrow o(x) = o(y)$ and $\pi^q_{xy} = \mathrm{id}_{o(x)+1}$, and
  - if $o(y) > \bar{\tau}$ then $x \mathrel{|\!\!\prec}^q y \leftrightarrow \exists\nu(o(x) = \bar{\tau} + \nu \wedge o(y) = \bar{\tau} + \omega + \nu)$ and
$$\pi^q_{xy}(\alpha) = \begin{cases} \alpha & \text{if } \alpha < \bar{\tau} \\ \bar{\tau} + \omega + \mu & \text{if } \alpha = \bar{\tau} + \mu. \end{cases}$$
- If $l(x) < \lambda^p < l(y) < \kappa$ then
$$x \mathrel{|\!\!\prec}^q y \leftrightarrow \exists z(l(z) = \lambda^p \wedge x \mathrel{|\!\!\prec}^q z \mathrel{|\!\!\prec}^q y),$$
and $\pi^q_{xy} = \pi^q_{zy} \circ \pi^q_{xz}$.
- If $\lambda^q = l(x) < l(y) = \kappa$ then $x \mathrel{|\!\!\prec}^q y \leftrightarrow f^q(o(x)) = o(y)$ and $\pi^q_{xy} = f^q \,|\, (o(x) + 1)$.
- If $l(x) < \lambda^q < l(y) = \kappa$ then
$$x \mathrel{|\!\!\prec}^q y \leftrightarrow \exists z(l(z) = \lambda^q \wedge x \mathrel{|\!\!\prec}^q z \mathrel{|\!\!\prec}^q y),$$
and $\pi^q_{xy} = \pi^q_{zy} \circ \pi_{xz}$.

That completes the definition of q. It is now a simple matter to check that the axioms for a morass condition hold for q; indeed, it is an identical verification process to that for the r of Lemma 51.

So q is a condition with $S^q = S^p \cup ((\sigma + \omega) \setminus \sigma)$, and by construction q extends p. ⊣

Lemma 54. *For any $p \neq \mathbf{1} \in \mathbb{P}$ there is a $q \leq p$ such that $S^q = S^p$, $\theta^q_{\lambda^q} = ot(S^q)$, and $\lambda^q = \lambda^p + \omega^{ot(S^q)}$.*

Proof. The proof is another variant of the same argument, this one being particularly straightforward. Let S^q and λ^q be as in the statement, set

$$\theta^q_\alpha = \begin{cases} \theta^p_\alpha & \text{if } \alpha \leq \lambda^p \\ \{\nu \,|\, \exists\zeta(\beta = \omega^\nu \cdot \zeta\} & \text{if } \alpha = \lambda^p + \beta,\ 0 < \beta < \omega^{\mathrm{ot}(S^q)} \\ \mathrm{ot}(S^q) & \text{if } \alpha = \lambda^q, \end{cases}$$

and let

$$\mathcal{S}^q = \left(\bigcup\{\{\alpha\} \times \theta^q_\alpha \mid \alpha \leq \lambda^q\}\right) \cup (\{\kappa\} \times S^q).$$

As always, take f^q to be the order preserving bijection from $\mathrm{ot}(S^q)$ to S^q. Define $x \mathrel{\vdash\!\!\!<}^q y$, and for x and y such that $x \mathrel{\vdash\!\!\!<}^q y$, π^q_{xy}, by the following cases. Again, id_α denotes the identity function on α.

- If $l(x) \geq l(y)$ then $x \not\mathrel{\vdash\!\!\!<}^q y$
- If $l(x) < l(y) \leq \lambda^p$, then $x \mathrel{\vdash\!\!\!<}^q y \leftrightarrow x \mathrel{\vdash\!\!\!<}^p y$, and $\pi^q_{xy} = \pi^p_{xy}$.
- If $\lambda^p \leq l(x) < l(y) < \kappa$, then $x \mathrel{\vdash\!\!\!<}^q y \leftrightarrow o(x) = o(y)$, and $\pi^q_{xy} = \mathrm{id}_{o(x)+1}$.
- If $l(x) < \lambda^p < l(y) < \kappa$ then

  $$x \mathrel{\vdash\!\!\!<}^q y \leftrightarrow \exists z(l(z) = \lambda^p \wedge x \mathrel{\vdash\!\!\!<}^q z \mathrel{\vdash\!\!\!<}^q y),$$

  and $\pi^q_{xy} = \pi^q_{zy} \circ \pi^q_{xz}$.
- If $\lambda^q = l(x) < l(y) = \kappa$ then $x \mathrel{\vdash\!\!\!<}^q y \leftrightarrow f^q(o(x)) = o(y)$ and $\pi^q_{xy} = f^q \mid (o(x) + 1)$.
- If $l(x) < \lambda^q < l(y) = \kappa$ then

  $$x \mathrel{\vdash\!\!\!<}^q y \leftrightarrow \exists z(l(z) = \lambda^q \wedge x \mathrel{\vdash\!\!\!<}^q z \mathrel{\vdash\!\!\!<}^q y),$$

  and $\pi^q_{xy} = \pi^q_{zy} \circ \pi_{xz}$.

It is routine to check that with this definition q is a morass condition and $q \leq p$. ⊣

Now let G be $\mathbb{P}$-generic over V. Let $M^G = \langle \mathcal{S}^G, \mathrel{\vdash\!\!\!<}^G, \langle \pi^G_{xy} \rangle_{x \mathrel{\vdash\!\!\!<}^G y} \rangle$ be the direct limit of the system G in $V[G]$. That is, $\mathcal{S}^G$ is the union over $p \in G$ of $\mathcal{S}^p$, for x and y in $\mathcal{S}^G$ we set $x \mathrel{\vdash\!\!\!<}^G y$ if and only if there is some condition $p \in G$ containing x and y such that $x \mathrel{\vdash\!\!\!<}^p y$, and in this case $\pi^G_{xy} = \pi^p_{xy}$.

Theorem 55. *With M^G defined as above, M^G is a mangrove in $V[G]$.*

Proof. Clearly $\mathcal{S}^G$ is a subset of $(\kappa \times \kappa) \cup (\{\kappa\} \times \kappa^+)$ and the maps π^G_{xy} are defined when they ought to be with the right domain and range. The relation $\mathrel{\vdash\!\!\!<}^G$ is tree relation since $\mathrel{\vdash\!\!\!<}^p$ is a tree relation for each condition p, and below level κ extensions of a condition only end-extend the relation. From Lemma 53, we have that $D_\tau = \{p \in \mathbb{P} \mid \tau \in S^p\}$ is dense in $\mathbb{P}$ for each $\tau < \kappa^+$, and so it follows that $G \cap D_\tau \neq \emptyset$ and hence $S^G = \kappa^+$. By Lemma 54 along with the κ-closure property of $\mathbb{P}$, it also follows that for any $\alpha < \kappa$, it is dense for the condition p to have $\lambda^p > \alpha$. In particular, θ^G_α is an ordinal greater than 0 for every $\alpha \leq \kappa$, and the full strength of the Left-alignment axiom is satisfied.

Monotonicity, Commutativity, M.1 and M.2 hold in M^G because they do in each condition. Similarly, to show that M.3 and M.4 hold in M^G, it suffices to show that they do for $y \in M^G$ with level κ. So suppose $y \in M^G$ and $l(y) = \kappa$. Given $\alpha < \kappa$, we may take a condition p such that $y \in p$ and $\lambda^p > \alpha$. Then there is some $z \in p$ with $l(z) = \lambda^p$ such that $z \mathrel{\vert\!\!\prec}^p y$. Hence, the set $\{l(z) \,|\, z \mathrel{\vert\!\!\prec}^G y\}$ is unbounded in κ, as required for M.4. Also, since the corresponding set in each condition is closed in κ, and the top levels λ^p of the conditions are unbounded in κ, we also have M.3: that the set is closed in κ.

Axiom M.5 is also immediate everywhere in M^G except level κ, from the fact that it holds in each condition. So suppose $y \in M^G$ has $l(y) = \kappa$, and let $\alpha < o(y)$. Let $w = \langle \kappa, \alpha \rangle$, and let $p \in G$ be such that $w, y \in p$. Let $v, x \in p$ be such that $l(v) = l(x) = \lambda^p$, $v \mathrel{\vert\!\!\prec}^p w$ and $x \mathrel{\vert\!\!\prec}^p y$. Then by the remarks following definition 40, $\pi^p_{xy}(o(v)) = \alpha$. Thus, M.5 holds in M^G at level κ as well.

Axioms M.6 and M.7 require more work to verify, but fortunately the situation is the same as in Velleman's paper [16]. We follow the arguments there, but recast them as transfinite inductions on natural well orders of the elements of M^G, rather than proofs by contradiction for the least counterexample.

To verify M.6, suppose that $x \mathrel{\vert\!\!\prec}^G y \in M^G$ with $o(x)$ a limit ordinal, and that M.6 holds for all such $x' \mathrel{\vert\!\!\prec}^G y'$ with y' lexicographically less than y; that is, $l(y') < l(y)$ or $l(y') = l(y) \wedge o(y') < o(y)$. Let $\zeta = \sup(\pi^G_{xy}``o(x))$ and let $z = \langle l(y), \zeta \rangle$. Clearly if $z = y$ we are done, so suppose $\zeta < o(y)$.

If y is a limit point of $\mathrel{\vert\!\!\prec}^G$, then M.5 holds of y and there is some $w \in M^G$ and $\mu < o(w)$ such that $x \mathrel{\vert\!\!\prec}^G w \mathrel{\vert\!\!\prec}^G y$ and $\pi_{wy}(\mu) = \zeta$. Denote $\langle l(w), \mu \rangle$ by u; then by axiom M.2, $u \mathrel{\vert\!\!\prec}^G z$, and $\pi^G_{uz} = \pi^G_{wy} \restriction (\mu + 1)$. By the composition axiom and M.1, we see that $\pi^G_{xw}``o(x) \subset \mu$, and moreover since $\pi^G_{xy}``o(x)$ is unbounded in $\zeta = \pi^G_{wy}(\mu)$, $\pi^G_{xw}``o(x)$ must be unbounded in μ. But now by the induction hypothesis, $x \mathrel{\vert\!\!\prec}^G u$ with $\pi^G_{xu} \restriction o(x) = \pi^G_{xw} \restriction o(x)$. Hence composing, we have that $x \mathrel{\vert\!\!\prec}^G z$ and

$$\pi^G_{xz} \restriction o(x) = \pi^G_{uz} \circ \pi^G_{xu} \restriction o(x) = \pi^G_{wy} \restriction (\mu + 1) \circ \pi^G_{xw} \restriction o(x) = \pi^G_{xy} \restriction o(x)$$

as required.

On the other hand, if y is not a limit point of $\mathrel{\vert\!\!\prec}^G$, we may take a $w \in M^G$ such that $w \mathrel{\vert\!\!\prec}^G_i y$. Note that by M4, $l(y) \neq \kappa$ in this case. If $w = x$ then M.6 for x and y follows from the fact that it holds in any condition in G containing both x and y. So suppose that $x \neq w$ and hence $x \mathrel{\vert\!\!\prec}^G w \mathrel{\vert\!\!\prec}^G_i y$. Let

$\mu = \sup(\pi^G_{xw}\text{``}o(x))$ and $u = \langle l(w), \mu\rangle$, so that by the induction hypothesis, $x \ltimes^G u$ with $\pi^G_{xu} \restriction o(x) = \pi^G_{xw} \restriction o(x)$. Note that it is possible that $u = w$. Let $\nu = \pi_{wy}(\mu)$ and $v = \langle l(y), \nu\rangle$; then if $\mu < o(w)$, $u \ltimes^G v$ with $\pi^G_{uv} = \pi^G_{wy} \restriction (\mu + 1)$ by M.2, and otherwise $u = w$, $v = y$ and the same statement trivially holds true. But now since $\pi^G_{xw}\text{``}o(x)$ is by definition unbounded in μ,

$$\begin{aligned}\sup(\pi^G_{uv}\text{``}\mu) &= \sup(\pi^G_{uv} \circ \pi^G_{xw}\text{``}o(x)) \\ &= \sup(\pi^G_{wy} \restriction (\mu+1) \circ \pi^G_{xw}\text{``}o(x)) \\ &= \sup(\pi^G_{xy}\text{``}o(x)) \\ &= \zeta\end{aligned}$$

If $\mu < o(w)$ and $\nu < o(y)$ we have by the induction hypothesis that $u \ltimes^G z$ and $\pi^G_{uz} \restriction \mu = \pi^G_{uv} \restriction \mu$. Otherwise, $u = w$ and $v = y$, so that $u \ltimes^G_i v$ and we also obtain $u \ltimes^G z$ and $\pi^G_{uz} \restriction \mu = \pi^G_{uv} \restriction \mu$, this time from the fact that it holds in any condition containing both u and v. So then $x \ltimes^G z$, and

$$\pi^G_{xz} \restriction o(x) = \pi^G_{uz} \circ \pi^G_{xu} \restriction o(x) = \pi^G_{uv} \circ \pi^G_{xw} \restriction o(x) = \pi^G_{wy} \circ \pi^G_{xw} \restriction o(x) = \pi^G_{xy} \restriction o(x).$$

Thus, we have by transfinite induction on the lexicographic order for y that M.6 holds of M^G.

To verify M.7 for M^G we will induct simply on $l(y)$. Let x, y and α be as in the antecedent of M.7, and suppose first that there is some w such that $x \ltimes^G w \ltimes^G y$ and $l(w) > \alpha$. This is in particular the case when y is a limit node of the tree, thanks to Monotonicity and M.3. For each $\nu < o(x)$, we have from M.2 that

$$\langle l(x), \nu\rangle \ltimes^G \langle l(w), \pi^G_{xw}(\nu)\rangle \ltimes^G \langle l(y), \pi^G_{wy} \circ \pi^G_{xw}(\nu)\rangle = \langle l(y), \pi_{xy}(\nu)\rangle.$$

Thus, since $\pi^G_{xy}\text{``}o(x)$ is unbounded in $o(y)$, it must be the case that $\pi^G_{xw}\text{``}o(x)$ is unbounded in $o(w)$. But now x, w and α fit the hypotheses for M.7, and $l(w) < l(y)$, so by the induction hypothesis, there is a γ such that $\langle \alpha, \gamma\rangle \ltimes^G w \ltimes^G y$.

So now suppose we are in the other situation, where there is some w with $l(w) \leq \alpha$ such that $w \ltimes^G_i y$. If $l(w) = \alpha$ we are clearly done. Also, if w were x, then M.7 would hold for x and y since it does for any condition containing them, and that would contradict the assumption that α is as in the hypothesis for that axiom. So suppose that $x \ltimes^G w \ltimes^G_i y$ with $l(w) < \alpha$; we will derive a similar contradiction in this case. As in the above case, we have for each $\nu < o(x)$ that

$$\langle l(x), \nu\rangle \ltimes^G \langle l(w), \pi^G_{xw}(\nu)\rangle \ltimes^G \langle l(y), \pi^G_{xy}(\nu)\rangle,$$

and π^G_{xw}"$o(x)$ is unbounded in $o(w)$. Let us denote by γ_ν that ordinal such that $\langle\alpha,\gamma_\nu\rangle \ltimes^G \langle l(y),\pi_{xy}(\nu)\rangle$, as we are assuming exists. For simplicity, let us also define notation for $\langle l(x),\nu\rangle$, $\langle l(w),\pi^G_{xw}(\nu)\rangle$, and $\langle l(y),\pi^G_{xy}(\nu)\rangle$ for any given $\nu < o(x)$: call them x_ν, w_ν and y_ν respectively. Then our string of tree relations can be rewritten and extended thus:

$$x_\nu \ltimes^G w_\nu \ltimes^G \langle\alpha,\gamma_\nu\rangle \ltimes^G y_\nu.$$

Now, given $\mu < o(w)$, take some $\nu < o(x)$ such that $\pi^G_{xw}(\nu) > \mu$. By M.2, we have that

$$\langle l(w),\mu\rangle \ltimes^G \langle\alpha,\pi^G_{w_\nu\langle\alpha,\gamma_\nu\rangle}(\mu)\rangle \ltimes^G \langle l(y),\pi^G_{w_\nu y_\nu}(\mu)\rangle,$$

and the corresponding functions π are the restriction of $\pi^G_{w_\nu\langle\alpha,\gamma_\nu\rangle}$ to $\mu+1$ and the restriction of $\pi^G_{\langle\alpha,\gamma_\nu\rangle y_\nu}$ to $\pi^G_{w_\nu\langle\alpha,\gamma_\nu\rangle}(\mu)+1$. We now have that the hypotheses for M.7 hold of w, y and α, and $w \ltimes^G_i y$, so the conclusion also obtains, since it does in any condition containing w and y. Of course, an ordinal γ given by M.7 for w, y, and α would also act as a witness to the truth of M.7 for x, y, and α. But in fact, we have a contradiction (as before) to the assumption that $w \ltimes^G_i y$, and so the supposed situation of having w satisfying $x \ltimes^G w \ltimes^G_i y$ and $l(w) < \alpha$ cannot occur. In any case, we may conclude that M.7 holds in M^G.

That completes the verification that M^G is a morass. To confirm that it is a mangrove, note that for any $p \in G$, λ^p is a mangal of M^G: this is immediate from the mangal requirement for extension. Thus, since the λ^p for $p \in G$ are unbounded in κ, M^G is a mangrove, as claimed. ⊣

3.3 A mangrove from a morass

In this section we deviate from the main programme of the chapter in order to answer a natural question that arose earlier — whether the existence of a morass implies the existence of a mangrove. A positive solution is provided by the theorems of [16], in which Velleman proves the equivalence of the existence of a morass with a certain forcing axiom. We shall show that this forcing axiom is sufficient to generate a mangrove from the mangrove forcing $\mathbb{P}$ defined in the previous section. Indeed, our argument will closely follow Velleman's argument that the forcing axiom gives a morass from his partial order.

We start with some preliminary definitions. Let P be a partial order, and let $\mathcal{D} = \{D_\alpha \,|\, \alpha < \kappa^+\}$ be an indexed family of open dense sets in P. For each $p \in P$, define the *realm* of p, $\mathrm{rlm}(p)$, to be the set $\{\alpha < \kappa^+ \,|\, p \in D_\alpha\}$.

For each $\alpha < \kappa^+$ let $P_\alpha = \{p \in P \mid \mathrm{rlm}(p) \subseteq \alpha\}$, and let $P^* = \bigcup_{\alpha<\kappa} P_\alpha$, the set of conditions with realms bounded below κ.

Definition 56 (Definition 1.1.2 of [16]). *A family* $\mathcal{D} = \{D_\alpha \mid \alpha < \kappa^+\}$ *of subsets of* P *is said to be* almost κ-indiscernible *if*

I.1 $P^* \neq \emptyset$*, and for all* $\alpha < \kappa$*,* $D_\alpha \cap P^*$ *is open dense in* P^**, and*

I.2 *for all* $\alpha < \kappa$*,* P_α *is* κ*-directed-closed,*

and there is a function σ *assigning to each SLOOP function* $f : \alpha \to \gamma$ *with* $\alpha < \kappa$ *and* $\gamma < \kappa^+$ *a function* $\sigma_f : P_\alpha \to P_\gamma$ *such that the following properties also hold for all such* f*.*

I.3 σ_f *is order preserving. That is,*

$$\forall p, q \in P_\alpha (p \leq q \to \sigma_f(p) \leq \sigma_f(q)).$$

I.4 *For all* $p \in P_\alpha$*,* $rlm(\sigma_f(p)) = f``rlm(p)$*.*

I.5 *If* $\gamma < \kappa$*, and for some* $\beta < \alpha$*,* $f \restriction \beta = id_\beta$ *and* $f(\beta) \geq \alpha$*, then for all* $p \in P_\alpha$*,* p *and* $\sigma_f(p)$ *are compatible in* P^**.*

I.6 *If* $f_1 : \alpha_1 \to \alpha_2$ *and* $f_2 : \alpha_2 \to \gamma$ *are strictly order preserving with* $\alpha_1, \alpha_2 < \kappa$ *and* $\gamma < \kappa^+$*, then*

$$\sigma_{f_2 \circ f_1} = \sigma_{f_2} \circ \sigma_{f_1}.$$

One of the main results of [16] is the following.

Theorem 57 (Theorems 1.1.3 and 2.1.6 of [16]). *The following are equivalent:*

1. *There exists a* $(\kappa, 1)$*-morass.*
2. *For any partial order* P *and* $\mathcal{D}$ *an almost* κ*-indiscernible family of subsets of* P*, there is a filter* G *in* P *such that for every* $D_\alpha \in \mathcal{D}$*,* $G \cap D_\alpha \neq \emptyset$*.*

Velleman further strengthens (2) above by making G hit κ-many additional sets E_ζ; the reader trying to reconcile the statements here and in [16] may take each E_ζ to be P. In this way statement (2) above can be seen to lie in strength between the formulations in Theorems 1.1.3 and 2.1.6 of [16], which are shown there to actually be equivalent.

To show that the existence of a morass implies the existence of a mangrove, we will show that the subsets of the mangrove forcing

$$D_\alpha = \{p \in \mathbb{P}_\kappa \,|\, \alpha \in S^p\}$$

for $\alpha < \kappa^+$ form an almost κ-indiscernible family. The filter we then obtain from Theorem 57 will have a mangrove as its direct limit, using essentially the same proof as for Theorem 55, where the filter in question was $\mathbb{P}$-generic over V.

Observe that with with these sets D_α, $\mathrm{rlm}(p) = S^p$ for all $p \in \mathbb{P}_\kappa$, and P_α as defined before Definition 56 is simply the set of all morass conditions p with $S^p \subseteq \alpha$. That is, $P_\alpha = \mathbb{P}_{\kappa,\alpha}$ as in Definition 49; we shall stick with the $\mathbb{P}_{\kappa,\alpha}$ notation here. Also, recall that for any morass condition p, $|S^p| \leq \theta_{\lambda^p} < \kappa$, so if $p \in \mathbb{P}_{\kappa,\kappa}$, then there is some $\alpha < \kappa$ such that $p \in \mathbb{P}_{\kappa,\alpha}$. Thus, $P^* = \mathbb{P}_{\kappa,\kappa}$; again, we will stick with the notation from Definition 49.

Proposition 58. *The sequence $\mathcal{D} = \langle D_\alpha \,|\, \alpha < \kappa^+\rangle$ as defined above is an almost κ-indiscernible family of subsets of the mangrove forcing $\mathbb{P}_\kappa$.*

Proof. Lemma 44 shows that $\mathbb{P}_{\kappa,\kappa}$ is non-empty, and Lemma 53 shows that for each $\alpha < \kappa$, $D_\alpha \cap \mathbb{P}_{\kappa,\kappa}$ is dense, with openness being clear from the definition of $\leq$ in $\mathbb{P}$. Therefore, I.1 holds. Also, I.2 is just Proposition 50 for $\alpha < \kappa$.

We must now define $\sigma_f : \mathbb{P}_{\kappa,\alpha} \to \mathbb{P}_{\kappa,\gamma}$ for f a SLOOP function with domain $\alpha < \kappa$ and codomain $\gamma < \kappa^+$. If $p \in \mathbb{P}_{\kappa,\alpha}$, let $\sigma_f(p)$ be that $q \in \mathbb{P}_{\kappa,\gamma}$ such that $\lambda^q = \lambda^p$, $q \overset{\lambda^p}{\sim} p$, and $S^q = f\text{``}S^p$, as given by Lemma 48. Of course, this definition makes I.4 and I.6 immediate. Further, the antecedent of I.5 makes Lemma 51 applicable, and the common extension constructed in that lemma clearly lies in $\mathbb{P}_{\kappa,\kappa}$, so I.5 also holds.

To check I.3, let g be a SLOOP function, and suppose $p \leq q$. Clearly λ^q remains a mangal of $\sigma_g(p)$, and we have $\sigma_g(q) \overset{\lambda^q}{\sim} \sigma_g(p)$. If we can now show that

$$\begin{aligned} &\mathrel{|\!\!<}^{\sigma_g(q)}\restriction \big((\{\lambda^q\} \times \theta_{\lambda^q}^{\sigma_g(q)}) \times (\{\kappa\} \times S^{\sigma_g(q)})\big) \\ &\qquad\qquad = \mathrel{|\!\!<}^{\sigma_g(p)}\restriction \big((\{\lambda^q\} \times \theta_{\lambda^q}^{\sigma_g(q)}) \times (\{\kappa\} \times S^{\sigma_g(q)})\big), \end{aligned}$$

and that for $x \mathrel{|\!\!<}^{\sigma_g(q)} y \in \mathcal{S}^{\sigma_g(q)}$ with $l(x) = \lambda^q$ and $l(y) = \kappa$, $\pi_{xy}^{\sigma_g(q)} = \pi_{xy}^{\sigma_g(p)}$, then we will be done, by Commutativity and the fact that $\lambda^q = \lambda^{\sigma_g(q)}$ is a mangal of $\sigma_g(q)$. Further, since $f^{\sigma_g(q)}$ is surjective onto $S^{\sigma_g(q)}$, it is sufficient to check that $\mathrel{|\!\!<}^{\sigma_g(q)} \subseteq \mathrel{|\!\!<}^{\sigma_g(p)}$ and $\pi_{xy}^{\sigma_g(q)} = \pi_{xy}^{\sigma_g(p)}$ on the relevant domain.

So suppose that $x \ltimes^{\sigma_g(q)} y$ for some $x, y \in \sigma_g(q)$ with $l(x) = \lambda^q$ and $l(y) = \kappa$. Note that by the definition of $\sigma_g(q)$, $f^{\sigma_g(q)} = g \circ f^q$, and likewise with q replaced by p. Now

$$x \ltimes^q \langle \kappa, f^q(o(x)) \rangle,$$

so

$$x \ltimes^p \langle \kappa, f^q(o(x)) \rangle.$$

Let z be that $z \in p$ such that $l(z) = \lambda^p$ and

$$x \ltimes^p z \ltimes^p \langle \kappa, f^q(o(x)) \rangle.$$

Then

$$x \ltimes^{\sigma_g(p)} z \ltimes^{\sigma_g(p)} \langle \kappa, g \circ f^q(o(x)) \rangle = \langle \kappa, f^{\sigma_g(q)}(o(x)) \rangle = y$$

as required.

Similarly,

$$\begin{array}{rcll} \pi^{\sigma_g(q)}_{xy} & = & f^{\sigma_g(q)} \restriction (o(x)+1) & \\ & = & g \circ f^q \restriction (o(x)+1) & \\ & = & g \circ \pi^q_{xy} & \\ & = & g \circ \pi^p_{xy} & \\ & = & g \circ f^p \circ \pi^p_{xz} & \text{for } z \text{ as above} \\ & = & f^{\sigma_g(p)} \circ \pi^p_{xz} & \\ & = & f^{\sigma_g(p)} \circ \pi^{\sigma_g(p)}_{xz} & \\ & = & \pi^{\sigma_g(p)}_{xy} & \end{array}$$

and we are done. ⊣

From Proposition 58 and Theorem 57, we obtain that if a morass exists at κ, then there is a filter G in $\mathbb{P}_\kappa$ intersecting every D_α.

In the proof of Theorem 55, the only dense sets G was required to meet were the sets D_α, along with the open dense set of conditions p with $\lambda^p > \gamma$, for each $\gamma < \kappa$. It turns out that using the following argument from [16] we can avoid directly requiring G to meet these latter sets.

Lemma 59. *If G is a filter over $\mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$ for each $\alpha < \kappa^+$, then $\{\lambda^p \,|\, p \in G\}$ is cofinal in κ.*

Proof. It will be convenient to work with the direct limit M^G of the filter G, with components $\mathcal{S}^G$, $\ltimes^G$ and π^G. Observe first that the verification of M.5 at level κ in M^G in the proof of Theorem 55 only required that G intersect every D_α, and thus goes through in our situation. We also have that in M^G, $\ltimes$ is a tree order and Monotonicity holds, since these properties are true in every condition, and below level κ the relation $\ltimes^p$ is only end-extended in moving to an extension of p. In particular, we have that for every $y \in \mathcal{S}^G$, $\{x \mid x \ltimes y\}$ contains at most one element from each level. Consider the node $y = \langle\kappa, \kappa\rangle$ in M^G. For every $x \ltimes^G y$, $|\pi^G_{xy}``o(x)| = |o(x)| < \kappa$. Thus, since κ is regular, there must be κ distinct $x \ltimes^G y$, and $\{l(x) \mid x \ltimes^G y\}$ is cofinal in κ. But now for each $x \in \mathcal{S}^G$, there is a $p \in G$ such that $x \in p$, and hence $l(x) \leq \lambda^p$. It follows that $\{\lambda^p \mid p \in G\}$ is cofinal in κ. ⊣

Theorem 60. *There is a morass at κ if and only if there is a mangrove at κ.*

Proof. Suppose there is a morass at κ. As mentioned above, by proposition 58 and Theorem 57, there is a filter G in $\mathbb{P}_\kappa$ intersecting every D_α. By Lemma 59, $\{\lambda^p \mid p \in G\}$ is cofinal in κ. Knowing this, the proof of Theorem 55 goes through for G, giving that the direct limit M^G is a mangrove. Of course, the other direction is trivial. ⊣

3.3.1 Aside: Complete embeddings

Although it is not necessary for the preceding arguments, the curious reader might wonder whether the inclusion maps $\mathbb{P}_{\kappa,\alpha} \hookrightarrow \mathbb{P}_\kappa$ are complete embeddings. This seemingly simple question actually takes quite some work to answer, so we here devote a subsection to its resolution.

Recall the following definition from [14].

Definition 61. *Let P and Q be partial orders. A function $i : P \to Q$ is a* complete embedding *if*

1. $\forall p, p' \in P\,(p' \leq p \to i(p') \leq i(p))$
2. $\forall p, p' \in P\,(p \perp p' \leftrightarrow i(p) \perp i(p'))$
3. $\forall q \in Q\, \exists p \in P\, \forall p' \in P\,(p' \leq p \to p' \parallel q)$

A condition p as in 3 is said to be a reduction *of q to P.*

Proposition 62. *For any limit ordinal $\alpha \leq \kappa^+$, the inclusion $\mathbb{P}_{\kappa,\alpha} \hookrightarrow \mathbb{P}_\kappa$ is a complete embedding if and only if cf$(\alpha) = \kappa$.*

Note that because of the requirement on morass conditions that S be closed under ordinal successors, $\mathbb{P}_{\kappa,\alpha+n} = \mathbb{P}_{\kappa,\alpha}$ for any limit ordinal α and natural number n. The proposition therefore effectively covers all cases.

Proof. Suppose first that $\alpha < \kappa$. Let $q \in \mathbb{P}_\kappa$ be a morass condition such that $\mathrm{ot}(S^q) > \alpha$ (such a q may be constructed using lemmas 44 and 53 and Proposition 50). Let $p \in \mathbb{P}_{\kappa,\alpha}$ be arbitrary; we shall show that p is not a reduction of q to $\mathbb{P}_{\kappa,\alpha}$.

Since $p \in \mathbb{P}_{\kappa,\alpha}$, $\mathrm{ot}(S^p) \leq \alpha$. Note that in the constructions for Lemma 54 and Proposition 50, the extension q obtained satisfies $\theta_{\lambda^q} = \mathrm{ot}(S^q)$. Thus, applying these results, we have that we can extend p to a $p' \in \mathbb{P}_{\kappa,\alpha}$ with $\lambda^{p'} > \lambda^q$ and $\theta^{p'}_{\lambda^{p'}} \leq \alpha$. We claim that p' is incompatible with q: if there were an $r \in \mathbb{P}_\kappa$ extending both p' and q, then $\lambda^{p'}$ would be a mangal of r, with $\theta^r_{\lambda^{p'}} = \theta^{p'}_{\lambda^{p'}} \leq \alpha$. But the edge $\langle \lambda^q, \alpha\rangle \mathrel{\vert\!\!\prec}^q \langle \kappa, f^q(\alpha)\rangle$ of q, and hence of r, cannot factor through such a level $\lambda^{p'}$ — as a consequence of M.1, $o(x) \leq o(y)$ for any $x \mathrel{\vert\!\!\prec}^r y$. Thus, no such common extension r can exist, and p is not a reduction of q to $\mathbb{P}_{\kappa,\alpha}$. Therefore the inclusion $\mathbb{P}_{\kappa,\alpha} \hookrightarrow \mathbb{P}_\kappa$ is not a complete embedding.

Suppose next that $\mathrm{cf}(\alpha) = \kappa$, and let q be any element of $\mathbb{P} \smallsetminus \mathbb{P}_{\kappa,\alpha}$. Let $S^q_\alpha = S^q \cap \alpha$; because $\mathrm{ot}(S^q_\alpha) \leq \mathrm{ot}(S^q) < \kappa$, S^q_α is bounded in α, and we will be able to overcome the problems seen in the $\alpha < \kappa$ case. Let $\tau = \sup(S^q_\alpha)$ and $\mu = \mathrm{ot}(S^q \smallsetminus S^q_\alpha)$. Using Lemma 48, let r be the condition such that $\lambda^r = \lambda^q$, $r \overset{\lambda^q}{\sim} q$, and $S^r = S^q_\alpha \cup ((\tau + \mu) \smallsetminus \tau)$. We claim that r is a reduction of q to $P_{\kappa,\alpha}$.

Suppose $p \leq r$ in $\mathbb{P}_{\kappa,\alpha}$, and note in particular that this implies that $S^p \supseteq S^r$. Applying Lemma 48 again, we can obtain a $p' \in \mathbb{P}_\kappa$ with $\lambda^{p'} = \lambda^p$, $p' \overset{\lambda^p}{\sim} p$, and $S^{p'} = (S^p \cap \tau) \cup (S^q \smallsetminus S^q_\alpha)$. Applying the construction of Lemma 51, we obtain a condition s extending both p and p', with $S^s = S^p \cup (S^q \smallsetminus S^q_\alpha)$. We claim that $s \leq q$. Certainly $\mathcal{S}^s$, and in particular S^s, is large enough. We also have that

$$q \overset{\lambda^q}{\sim} r \overset{\lambda^r}{\sim} p \overset{\lambda^p}{\sim} s$$

so $q \overset{\lambda^q}{\sim} s$. Since λ^q is a mangal of q, it suffices now to check that

$$\mathrel{\vert\!\!\prec}^q\restriction \big((\{\lambda^q\} \times \theta^q_{\lambda^q}) \times (\{\kappa\} \times S^q)\big) = \mathrel{\vert\!\!\prec}^s\restriction \big((\{\lambda^q\} \times \theta^q_{\lambda^q}) \times (\{\kappa\} \times S^q)\big),$$

and that for $x \mathrel{\vert\!\!\prec}^q y \in \mathcal{S}^q$ with $l(x) = \lambda^q$ and $l(y) = \kappa$, $\pi^q_{xy} = \pi^s_{xy}$.

Now for x and y in q with $l(x) = \lambda^q$ and $l(y) = \kappa$, $x \mathrel{\vert\!\!\prec}^q y$ if and only if $f^q(o(x)) = o(y)$, and $\pi^q_{xy} = f^q \restriction (o(x) + 1)$. It therefore suffices to show that for such x and y, $x \mathrel{\vert\!\!\prec}^s y$ whenever $f^q(o(x)) = o(y)$, with $\pi^s_{xy} = f^q \restriction (o(x)+1)$ — the converse holds because f^q, the order preserving bijection from $\mathrm{ot}(S^q)$ to S^q, is surjective onto S^q. So suppose $x \mathrel{\vert\!\!\prec}^q y$, that is, $f^q(o(x)) = o(y)$. There are two cases to consider.

- If $o(x) < \mathrm{ot}(S^q_\alpha)$, then x and y are actually in r, and we have $x \mathrel{\vert\!\!\prec}^r y$ with $\pi^r_{xy} = \pi^q_{xy}$. Since $r \geq p \geq s$, we have $x \mathrel{\vert\!\!\prec}^s y$ with $\pi^s_{xy} = \pi^q_{xy}$.

- If $\mathrm{ot}(S^q_\alpha) \leq o(x) < \mathrm{ot}(S^q)$, let β be such that $o(x) = \mathrm{ot}(S^q_\alpha) + \beta$. Then in r,

  $$x \mathrel{|\!\!\prec}^r \langle \kappa, \tau + \beta \rangle, \text{ with } \pi^r_{x\langle \kappa, \tau+\beta \rangle}(\gamma) = \begin{cases} f^q(\gamma) & \text{if } \gamma < \mathrm{ot}(S^q_\alpha) \\ \tau + \delta & \text{if } \gamma = \mathrm{ot}(S^q_\alpha) + \delta, \delta < \beta, \end{cases}$$

  and so the same also holds true with r replaced everywhere by p. We therefore have that

  $$x \mathrel{|\!\!\prec}^p \langle \lambda^p, \mathrm{ot}(S^p \cap \tau) + \beta \rangle \mathrel{|\!\!\prec}^p \langle \kappa, \tau + \beta \rangle,$$

  and

  $$\pi^p_{x\langle \lambda^p, \mathrm{ot}(S^p \cap \tau) + \beta \rangle} = (f^p)^{-1} \circ \pi^p_{x\langle \kappa, \tau+\beta \rangle}.$$

  Recalling that $p' \overset{\lambda^p}{\sim} p$ and considering $f^{p'}$, we may conclude that

  $$x \mathrel{|\!\!\prec}^{p'} \langle \lambda^p, \mathrm{ot}(S^p \cap \tau) + \beta \rangle \mathrel{|\!\!\prec}^{p'} \langle \kappa, f^q(\mathrm{ot}(S^q_\alpha) + \beta) \rangle = y,$$

  with

  $$\pi^{p'}_{xy} = f^{p'} \circ (f^p)^{-1} \circ \pi^p_{x\langle \kappa, \tau+\beta \rangle}.$$

  If $\gamma < \mathrm{ot}(S^q_\alpha)$, then

  $$\pi^{p'}_{xy}(\gamma) = f^{p'} \circ (f^p)^{-1}(f^q(\gamma)) = f^q(\gamma).$$

  If $\gamma = \mathrm{ot}(S^q_\alpha) + \delta$, then

  $$\begin{aligned} \pi^{p'}_{xy}(\gamma) &= f^{p'} \circ (f^p)^{-1}(\tau + \delta) \\ &= f^{p'}(\mathrm{ot}(S^p \cap \tau) + \delta) \\ &= f^q(\mathrm{ot}(S^q_\alpha) + \delta) \\ &= f^q(\gamma). \end{aligned}$$

  Hence, we have $x \mathrel{|\!\!\prec}^{p'} y$ with $\pi^{p'}_{xy} = f^q \restriction (o(x) + 1) = \pi^q_{xy}$, and therefore the same is true in s.

In either case $s \leq q$ as claimed, and so r is indeed a reduction of q to $\mathbb{P}_{\kappa,\alpha}$.

Of course there is more to complete embeddings than the existence of reductions. The inclusion $\mathbb{P}_{\kappa,\alpha} \hookrightarrow \mathbb{P}$ certainly preserves the relation $\leq$; we must also show that it preserves incompatibility. So suppose p and q in $\mathbb{P}_{\kappa,\alpha}$ are compatible in $\mathbb{P}$ with some common extension $r \in \mathbb{P}$. Let r' be the unique condition such that $\lambda^{r'} = \lambda^r$, $r' \overset{\lambda^r}{\sim} r$, and $S^{r'} = S^r \cap \alpha$, as given by Lemma 48. Clearly $r' \in \mathbb{P}_{\kappa,\alpha}$, and it is straightforward to verify that because $p \leq r$

and $q \leq r$, $p \leq r'$ and $q \leq r'$. So the inclusion $\mathbb{P}_{\kappa,\alpha} \hookrightarrow \mathbb{P}$ indeed preserves incompatibility, and hence is a complete embedding.

Finally, suppose that $\alpha > \kappa$ but $\mathrm{cf}(\alpha) < \kappa$. Let $q \in \mathbb{P} \smallsetminus \mathbb{P}_{\kappa,\alpha}$ be such that $S^q \smallsetminus \alpha \neq \emptyset$ and $S^q \cap \alpha$ is cofinal in α (using Lemmas 44 and 53 and Proposition 50 as usual). We claim that q has no reduction to $\mathbb{P}_{\kappa,\alpha}$. Suppose for the sake of contradiction that p is a reduction of q to $\mathbb{P}_{\kappa,\alpha}$, and let r be a common extension of p and q in $\mathbb{P}$. As in the argument above for incompatibility preservation, let r' be the condition obtained from r but truncating S^r to $S^r \cap \alpha$. We again obtain $r' \leq p$, so r' is also a reduction of q to $\mathbb{P}$. Also, if q' is likewise obtained from q by truncating S^q to $S^q \cap \alpha$, then it is straightforward to check that $q' \geq r'$ (indeed, $q' \geq q \geq r$, so the verification for incompatibility preservation also applies to this case). In particular, $S^{r'} \supseteq S^q \cap \alpha$, and for any $\langle \lambda^q, \delta \rangle$ in q with $\delta < \mathrm{ot}(S^q \cap \alpha)$, $\langle \lambda^q, \delta \rangle \mathrel{\vert\!\!<}^{r'} \langle \kappa, f^q(\delta) \rangle$.

Let r'' be obtained from r' by applying Lemma 54, so that in particular $\theta^{r''}_{\lambda^{r''}} = \mathrm{ot}(S^{r''})$ and $S^{r''} = S^{r'}$. Since the set of $f^q(\delta)$ for $\delta < \mathrm{ot}(S^q \cap \alpha)$ is cofinal in $\alpha = \sup(S^{r'})$, it follows that the set

$$\big\{\zeta \mid \exists \delta < \mathrm{ot}(S^q \cap \alpha)\big(\langle \lambda^q, \delta \rangle \mathrel{\vert\!\!<}^{r''} \langle \lambda^{r''}, \zeta \rangle \mathrel{\vert\!\!<}^{r''} \langle \kappa, f^q(\delta) \rangle\big)\big\}$$

is unbounded in $\theta^{r''}_{\lambda^{r''}}$.

Now suppose that s is a common extension of q and r'' with $\lambda^s > \lambda^{r''}$ (using Lemma 54 again if necessary). Let $\beta = \mathrm{ot}(S^q \cap \alpha)$; then there is some $\tau \geq \alpha$ such that $\langle \lambda^q, \beta \rangle \mathrel{\vert\!\!<}^q \langle \kappa, \tau \rangle$, and hence $\langle \lambda^q, \beta \rangle \mathrel{\vert\!\!<}^s \langle \kappa, \tau \rangle$. Since $\lambda^{r''}$ is a mangal of s, let γ be such that $\langle \lambda^q, \beta \rangle \mathrel{\vert\!\!<}^s \langle \lambda^{r''}, \gamma \rangle \mathrel{\vert\!\!<}^s \langle \kappa, \tau \rangle$. We must have $\gamma < \theta^s_{\lambda^{r''}} = \theta^{r''}_{\lambda^{r''}}$, so there is some $\delta < \mathrm{ot}(S^q \cap \alpha)$ and $\zeta > \gamma$ such that $\langle \lambda^q, \delta \rangle \mathrel{\vert\!\!<}^{r''} \langle \lambda^{r''}, \zeta \rangle \mathrel{\vert\!\!<}^{r''} \langle \kappa, f^q(\delta) \rangle$. Now

$$\pi^s_{\langle \lambda^{r''}, \gamma \rangle \langle \kappa, \tau \rangle} \circ \pi^s_{\langle \lambda^q, \beta \rangle \langle \lambda^{r''}, \gamma \rangle}(\delta) = \pi^s_{\langle \lambda^q, \beta \rangle \langle \kappa, \tau \rangle}(\delta) = \pi^q_{\langle \lambda^q, \beta \rangle \langle \kappa, \tau \rangle}(\delta) = f^q(\delta)$$

So by axiom M.2, $\langle \lambda^{r''}, \pi^s_{\langle \lambda^q, \beta \rangle \langle \lambda^{r''}, \gamma \rangle}(\delta) \rangle \mathrel{\vert\!\!<}^s \langle \kappa, f^q(\delta) \rangle$. But $\pi^s_{\langle \lambda^q, \beta \rangle \langle \lambda^{r''}, \gamma \rangle}(\delta) < \gamma < \zeta$, and we already have that $\langle \lambda^{r''}, \zeta \rangle \mathrel{\vert\!\!<}^s \langle \kappa, f^q(\delta) \rangle$, giving a contradiction. Thus, there cannot be a reduction p of q to $\mathbb{P}_{\kappa,\alpha}$, and hence the inclusion $\mathbb{P}_{\kappa,\alpha} \hookrightarrow \mathbb{P}$ is not a complete embedding. $\dashv$

3.4 Homogeneity

When we force to obtain a morass in the presence of various large cardinal axioms, we will sometimes be able to preserve the large cardinal property by simply requiring that our generic be below some master condition. Of course, for each cardinal κ with the given large cardinal property, there will be a different master condition. Hence, to be able to preserve the large cardinal property for all of these cardinals at once, one would like to be able to impose each master condition "after the fact", substituting it into the "bottom" of any given generic to give a new generic suited to the large cardinal at hand. This is achieved with Theorem 72 of this section.

In some forcing posets, such as the standard ones to force $\diamondsuit$ and $\square$, compatibility implies comparability, and further one of the conditions will be an initial segment of the other. When these initial segments may be freely interchanged giving new conditions, a great deal of homogeneity for the partial order results.

While we do not have these same properties in $\mathbb{P}$, our notion of μ-equivalence is an appropriate substitute for comparability which does follow from compatibility (Lemma 47), with the part of the morass condition below μ being a suitable "initial segment" for us to interchange. To be precise, we will use μ-equivalence in defining automorphisms on suitably chosen open dense suborders of $\mathbb{P}$, with the automorphism being a sort of "initial segment interchange". We will then show in Theorem 72 that such automorphisms may be used to construct, within an arbitrary generic extension $V[G]$, $\mathbb{P}$-generics over V containing any given condition. The definition of these automorphisms is the crux of our argument for Theorem 72, and by extension, the large cardinal preservation arguments in Sections 3.5 and 3.6. Velleman's partial order of [16], with no mangal requirement for extension, lacks this crucial homogeneity.

First, we define the suborders of interest.

Definition 63. *For any* $\alpha < \kappa$, *let*

$$\mathbb{P}^\alpha = \{p \in \mathbb{P} \mid \lambda^p \geq \alpha\}.$$

Proposition 64. *For all* $\alpha < \kappa$, $\mathbb{P}^\alpha$ *is open dense in* $\mathbb{P}$.

Proof. Density follows from Lemma 54 and Proposition 50, and openness is immediate from the definition of $\leq$. $\dashv$

Of course, this means that generics for $\mathbb{P}^\alpha$ are easily translatable into generics for $\mathbb{P}$ and conversely, while preserving the extension universe $V[G]$.

We will define our automorphisms by "interchanging" initial pieces of conditions.

Definition 65. *Let p, q and r in $\mathbb{P}$ be three conditions such that $\lambda^q \geq \lambda^p = \lambda^r$, $\theta^p_{\lambda^p} = \theta^r_{\lambda^r}$, and $q \overset{\lambda^p}{\sim} p$. Define q^p_r as follows: let*

$$\begin{aligned}
\theta^{q^p_r}_\alpha &= \begin{cases} \theta^r_\alpha & \text{if } \alpha \leq \lambda^p \\ \theta^q_\alpha & \text{if } \alpha \geq \lambda^p \end{cases} \\
S^{q^p_r} &= S^q
\end{aligned}$$

$$\begin{aligned}
x \mathrel{|\!\prec}^{q^p_r} y \quad \longleftrightarrow \quad & \big(l(x) < l(y) \leq \lambda^p \ \wedge \ x \mathrel{|\!\prec}^r y\big) \ \vee \\
& \big(\lambda^p \leq l(x) < l(y) \ \wedge \ x \mathrel{|\!\prec}^q y\big) \ \vee \\
& \big(l(x) < \lambda^p < l(y) \ \wedge \\
& \qquad \exists z \in \mathcal{S}^q(l(z) = \lambda^p \wedge x \mathrel{|\!\prec}^r z \mathrel{|\!\prec}^q y)\big),
\end{aligned}$$

and for $x \mathrel{|\!\prec}^{q^p_r} y$,

$$\pi^{q^p_r}_{xy} = \begin{cases} \pi^r_{xy} & \text{if } l(x) < l(y) \leq \lambda^p \\ \pi^q_{xy} & \text{if } \lambda^p \leq l(x) < l(y) \\ \pi^q_{zy} \circ \pi^r_{xz} & \text{if } l(z) = \lambda^p \wedge x \mathrel{|\!\prec}^r z \mathrel{|\!\prec}^q y. \end{cases}$$

Intuitively q^p_r is just q with p replaced by r, although S^p and S^r are irrelevant for the construction.

Proposition 66. *For p, q and r as in Definition 65, q^p_r is a morass condition and $q^p_r \overset{\lambda^p}{\sim} r$.*

Proof. It is straightforward to check that all of the requirements for a morass condition hold for q^p_r, because they hold for q and r. In particular, note that M.2 holds for $l(x) < \lambda^p < l(y)$ because $\pi^{q^p_r}_{xy}$ is defined by composition for such x and y. Also, because we only require M.6 and M.7 to hold for immediate $\mathrel{|\!\prec}$-successors in a morass condition, these two axioms are immediate from the fact that they hold in q and r. It is immediate from the definition that λ^p is also a mangal of the new condition q^p_r, and $\lambda^p = \lambda^r$ so it is also a mangal of r. Since the requisite equalities trivially hold, it follows that $q^p_r \overset{\lambda^p}{\sim} r$. $\dashv$

Definition 67. *Let p and r in $\mathbb{P}$ be such that $\lambda^p = \lambda^r$ and $\theta^p_{\lambda^p} = \theta^r_{\lambda^r}$. Then define $\varphi^{p,r} : \mathbb{P}^{\lambda^p} \to \mathbb{P}^{\lambda^p}$ by*

$$\varphi^{p,r}(q) = \begin{cases} q^p_r & \text{if } q \overset{\lambda^p}{\sim} p \\ q^r_p & \text{if } q \overset{\lambda^p}{\sim} r \\ q & \text{otherwise.} \end{cases}$$

Note first that this is well defined: if $p \overset{\lambda^p}{\sim} q \overset{\lambda^p}{\sim} r$, then $p \overset{\lambda^p}{\sim} r$, and it is easy to see that $q^p_r = q^r_p = q$.

Because of the mangal requirement for the relation $\overset{\lambda^p}{\sim}$, $\varphi^{p,r}$ is self-inverse: ${\varphi^{p,r}}^2(q) = q$ for all $q \in \mathbb{P}^{\lambda^p}$. Note that this is the point in the argument which necessitates the mangal requirement, and concomitantly where the definition of $\leq$ in Velleman's partial order fails for our purposes. We could have defined q^p_r without such a requirement, and would still have obtained a morass condition, but the corresponding φ would not have been surjective. The range of such a φ would not include any condition that was "the same as p up to level λ^p" but which did not have λ^p as a mangal, and likewise for r. Since the set of such conditions is not codense in $\mathbb{P}^{\lambda^p}$, this is a real cause for concern, hence the introduction of mangals.

Proposition 68. *For conditions p and r satisfying the requirements of Definition 67, the function $\varphi^{p,r}$ is an automorphism of the poset $\mathbb{P}^{\lambda^p}$.*

Proof. Since $\varphi^{p,r}$ is a self-inverse bijection, it only remains to show that $\varphi^{p,r}$ respects $\leq$. So suppose that $q \leq s \in \mathbb{P}^{\lambda^p}$. Then $q \overset{\lambda^s}{\sim} s$ and $\lambda^p < \lambda^s$, so $q \overset{\lambda^p}{\sim} p \leftrightarrow s \overset{\lambda^p}{\sim} p$ and $q \overset{\lambda^p}{\sim} r \leftrightarrow s \overset{\lambda^p}{\sim} r$ by Lemma 46 (note that this would not necessarily be the case if we were to use compatibility in place of the $\overset{\lambda^p}{\sim}$ relation).

For the sake of the argument at hand, we may therefore assume without loss of generality that $q \overset{\lambda^p}{\sim} p \overset{\lambda^p}{\sim} s$, so that $\varphi^{p,r}(q) = q^p_r$ and $\varphi^{p,r}(s) = s^p_r$. To see that λ^q remains a mangal of $\varphi^{p,r}(s)$, note by Proposition 66 that $\lambda^p = \lambda^r$ is a mangal of s^p_r, so any edge of s^p_r factors as an edge in r followed by an edge in s from level λ^p, the latter factorising through level λ^q since λ^q is a mangal of s. The remainder of the verification that $q^p_r \leq s^p_r$ is trivial from the definitions. ⊣

The following lemma will also be useful.

Lemma 69. *If p and r are conditions in $\mathbb{P}$ such that $\lambda^p = \lambda^r$ and $\theta^p_{\lambda^p} = \theta^r_{\lambda^r}$, and further $S^p = S^r$, then $\varphi^{p,r}(p) = r$ and $\varphi^{p,r}(r) = p$.*

Proof. Of course $\varphi^{p,r}$ is symmetric in p and r, so it suffices to consider one case. The set S^r determines f^r and hence also $\ltimes^r$ and π^r_{xy} up from level λ^r, and $\varphi^{p,r}(p) = p^p_r \overset{\lambda^r}{\sim} r$. ⊣

With these tools at our disposal, we are almost ready to prove the main theorem of this section. The idea of the proof of Theorem 72 is to find a condition r in G large enough to "cover" a given p, so that we can use $\varphi^{p,r}$ to "patch in" p in place of r in the generic. The following two lemmas allow us to find a suitable such r.

Lemma 70. *For any subset X of κ^+ such that $|X| < \kappa$, the set*

$$D_X = \{r \in \mathbb{P} \,|\, X \subset S^r\}$$

is open dense in $\mathbb{P}$.

Proof. Density is obvious by combining Lemma 53 with Proposition 50. Openness is immediate from the definition of extension. $\dashv$

Lemma 71. *For any ordinal $\mu < \kappa$, the set*

$$E_\mu = \{r \in \mathbb{P} \,|\, \exists \alpha > ot(S^r)(\lambda^r = \mu + \omega^{ot(S^r)} \cdot \alpha) \wedge \theta^r_{\lambda^r} = ot(S^r)\}$$

is dense in $\mathbb{P}$.

Proof. First note that μ only affects the minimum value of λ for elements of E_μ, and not its "form": it is a simple result of ordinal arithmetic, that for any μ, γ and α_0, there is an α_1 such that $\mu + \omega^\gamma \cdot \alpha_0 = \omega^\gamma \cdot \alpha_1$. Now given any p in $\mathbb{P}$, we may simply use Lemma 54 repeatedly, "gluing together" with Proposition 50 (noting that the argument there will not increase S), to give an extension in E_μ. $\dashv$

We are now ready to prove the main theorem of this section.

Theorem 72 (Homogeneity). *Let G be $\mathbb{P}$-generic over V and let p be a condition in $\mathbb{P}$. Then there exists a $G' \subset V[G]$ such that $p \in G'$ and G' is also $\mathbb{P}$-generic over V. Moreover, $V[G'] = V[G]$.*

Note that we write $G' \subset V[G]$ rather than $G' \in V[G]$ because we are not assuming that $\mathbb{P}$ is set sized from the perspective of the ground model V — this will be relevant in Section 3.6 below.

Proof. By Lemmas 70 and 71, there is some $r \in G \cap D_{S^p} \cap E_{\lambda^p}$. We claim that p may be extended to a condition p' such that $S^{p'} = S^r$, $\lambda^{p'} = \lambda^r$, and $\theta^{p'}_{\lambda^{p'}} = \theta^r_{\lambda^r}$. Note that in Lemma 53, the extension q of p constructed satisfies $\lambda^q = \lambda^p + \omega^{\mathrm{ot}(S^q)}$. This construction is to add a single new limit ordinal to S, whereas we wish to add the whole of $S^r \setminus S^p$. However, we may construct a sequence of conditions q_α, whereby if $\langle \sigma_\alpha \,|\, \alpha < \beta \rangle$ is the increasing enumeration of limit ordinals in $S^r \setminus S^p$, $q_{\alpha+1}$ is obtained from q_α by the method of Lemma 53 adding σ_α and its finitary successors, $q_0 = p$, and q_λ for limit ordinals λ is obtained by the method of Proposition 50 to close off with a new top level. Using Proposition 50 one last time if β is a limit ordinal and taking $q_{\beta-1}$ otherwise, we get a lower bound q for the sequence such that $S^q = S^r$ and

$$\lambda^q = \lambda^p + \sum_{0<\alpha<\beta} \omega^{\mathrm{ot}(S^{q_\alpha})}.$$

Now, the summation on the right (excluding λ^p) is clearly less than or equal to $\omega^{\mathrm{ot}(S^q)} \cdot \beta$. Further, β must be less than or equal to $\mathrm{ot}(S^q)$, so we have that

$$\lambda^q \leq \lambda^p + \omega^{\mathrm{ot}(S^q)} \cdot \mathrm{ot}(S^q).$$

But now

$$\lambda^r = \lambda^p + \omega^{\mathrm{ot}(S^r)} \cdot \delta = \lambda^p + \omega^{\mathrm{ot}(S^q)} \cdot \delta \quad \text{for some } \delta > \mathrm{ot}(S^q),$$

so noting as in Lemma 71 that adding $\omega^{\mathrm{ot}(S^q)}$ to an ordinal yields an ordinal divisible by $\omega^{\mathrm{ot}(S^q)}$, we see that we may extend q further to a p' with $S^{p'} = S^r$ and $\lambda^{p'} = \lambda^r$, by simply applying Lemma 54 and Proposition 50 repeatedly. Also, since we must use Lemma 54 at least once to get p', we will further have that $\theta^{p'}_{\lambda^{p'}} = \mathrm{ot}(S^{p'}) = \mathrm{ot}(S^r) = \theta^r_{\lambda^r}$, and so p' has all of the desired attributes in common with r.

Now in $V[G]$, let G^{λ^r} denote $G \cap \mathbb{P}^{\lambda^r}$, let $G^{\lambda^r}_{p'}$ denote $\varphi^{p',r}``G^{\lambda^r}$, and let

$$G' = \{q \in \mathbb{P} | \exists s \in G^{\lambda^r}_{p'} (s \leq q)\}.$$

Because $\varphi^{p',r}$ is a poset automorphism and $\mathbb{P}^{\lambda^r}$ is open dense in $\mathbb{P}$, G' is $\mathbb{P}$-generic over V, and $V[G'] = V[G]$. Because $\varphi^{p',r}(r) = p'$ and $r \in G^{\lambda^r}$, $p' \in G^{\lambda^r}_{p'}$, and so $p \in G'$, as desired. $\dashv$

3.5 Preserving large cardinals

In [9], Sy Friedman shows that using a reverse Easton iteration of forcings to construct morasses, one may force morasses to exist at every cardinal while preserving a hyperstrong or n-superstrong cardinal, for $0 < n \leq \omega$. However, except in the case of 1-superstrong cardinals, this result is only achieved for a single cardinal, as the generic used must fall below a specific master condition that is dependent on the embedding witnessing n-superstrength or hyperstrength, and so a single generic cannot be used to preserve the large cardinal strength of many different cardinals. On the other hand, with our Homogeneity Theorem (Theorem 72), we can circumvent this problem, expressing for each master condition p the extension $V[G]$ as $V[G']$ for some $G' \ni p$. We give the details of the argument in this section.

Assume the GCH holds (this can of course be forced while preserving large cardinals as in [8] and Chapter 2).

Definition 73. *The* Global Mangrove Partial Order *R is the reverse Easton iteration of partial orders $\dot{Q}_\alpha$, where $\dot{Q}_\alpha$ names $\mathbb{P}_\alpha$ if α is a regular uncountable cardinal, and $\dot{Q}_\alpha$ names the trivial forcing otherwise.*

Note that $\mathbb{P}_\kappa$ will continue to denote the forcing to add a mangrove at κ, whereas R_κ will now denote the iteration up to stage κ of such forcings.

Theorem 74. *If $V \models ZFC + GCH$ and G is R-generic over V, then*

$$V[G] \models ZFC + GCH + \text{“There is a mangrove at } \kappa \text{ for every regular cardinal } \kappa.\text{”}$$

Proof. For each regular cardinal κ, we have that $\mathbb{P}_\kappa$ is κ-closed and κ^+-cc (Propositions 50 and 52), and moreover $|\mathbb{P}_\kappa| = (\kappa^+)^\kappa = \kappa^+$. As mentioned in Subsection 3.2.4, this implies that forcing with $\mathbb{P}_\kappa$ preserves cardinals, and a nice names argument gives that the GCH is preserved as well.

Applying the Factor Lemma (Lemma 25), we may factorise R as $R_\kappa * \dot{R}^\kappa$ for any κ, with R^κ forced to be κ-closed. Proposition 26 then yields that R is tame.

We prove that cardinals and the GCH are preserved by induction on the iteration length. Successor stages are immediate from the observations about $\mathbb{P}_\kappa$ above.

At singular limit cardinal λ stages, we observe by factorising R_λ as $R_{\kappa^+} * R^{\kappa^+}$ that each cardinal $\kappa < \lambda$ is preserved, whence λ itself is also preserved, and moreover the GCH is preserved below λ. The iteration R_λ has a dense suborder of size $\prod_{\kappa<\lambda} \kappa^+ = \lambda^+$, so cardinals greater than λ^+ are preserved, and the GCH holds at and above $(\lambda^+)^V$. Finally, taking unions over $\kappa < \lambda$ of nice names for subset of κ, we obtain λ^+ nice names for subsets of λ, so there is a surjection from $(\lambda^+)^V$ to $(2^\lambda)^{V[G_\lambda]}$. Hence, λ^+ is also preserved, as is the GCH at κ.

At regular limit stages λ, where we take direct limits, the argument is similar but easier, since R_λ will have a dense suborder of size $\sum_{\kappa<\lambda} \kappa^+ = \lambda$. Indeed if λ is Mahlo, we even have that R_λ satisfies the λ chain condition — see [1], Corollary 2.4. In any case, we have shown that every stage of the iteration preserves cardinals and the GCH, and so since the closedness of R^κ increases with κ, R itself preserves cardinals and the GCH.

Showing that mangroves exist at every regular uncountable cardinal of the generic extension is now straightforward. Given a regular uncountable κ and factorising R as $R_\kappa * \dot{\mathbb{P}}_\kappa * \dot{R}^{\kappa+1}$, we have that there is a mangrove at κ after the first $\kappa + 1$ stages of the iteration. Now the statement “M is a mangrove at κ” is absolute for models containing M, κ as a regular cardinal, and κ^+ as its successor. Thus, since the final part $R^{\kappa+1}$ of the iteration is

κ^+-closed, the mangrove at κ will remain a mangrove after this final part of the iteration, and we are done. $\dashv$

Theorem 75. *Suppose that $V \models$ ZFC + GCH, and G is R-generic over V. Then for any $n \in \omega + 1$ and any cardinal κ, if*

$$V \models \kappa \textit{ is } n\textit{-superstrong}$$

then

$$V[G] \models \kappa \textit{ is } n\textit{-superstrong.}$$

Similarly, if κ is hyperstrong in V, then κ is hyperstrong in $V[G]$.

Proof. As mentioned above, the case of 1-superstrongs is proven in [8]. The proofs in the other cases also follow exactly as in that paper, with the simple modification that because of our homogeneity theorem, a generic $G' \subset V[G]$ with $V[G'] = V[G]$ may be chosen containing any given master condition. Now, we should be careful at this point, because for n-superstrength with $n > 1$, we will want to take a master condition at not just a single iterand of the forcing, but many. Factorise R as

$$R_\kappa * \dot{\mathbb{P}}_\kappa * R^{(\kappa,j(\kappa))} * \cdots * \mathbb{P}_{j^n(\kappa)} * R^{j^n(\kappa)+1},$$

and let G be R-generic over V. We wish to modify (within $V[G]$) the upper part of G so as to contain the master conditions for the lower part. There are no restrictions on $G_{j(\kappa)}$. We can modify G so that $G(j(\kappa))$ contains the master condition from $G(\kappa)$. Now $R^{(j(\kappa),j^2(\kappa))}$ is $j(\kappa)^+$-directed-closed since each of its iterands is (see [1], Theorem 2.7), so we may take a single master condition in $R^{(j(\kappa),j^2(\kappa))}$ determined by $G^{(\kappa,j(\kappa)}$. By similar arguments to those in [7], we may combine the automorphisms we have from Theorem 72 to give an automorphism for (a dense suborder of) $R^{(j(\kappa),j^2(\kappa))}$, each initial piece of which leaves later iterands $\dot{\mathbb{P}}_\alpha$ unchanged up to equality forced by $1_{R^{(j(\kappa),\alpha)}}$. This automorphism allows us to modify $G^{(j(\kappa),j^2(\kappa))}$, and hence G, to lie below the master condition. Repeating all of this another $n-2$ times, we construct a $G'_{j^{n-1}(\kappa)} \subset V[G]$ below all of the necessary master conditions to make the arguments of [8] go through. Of course, if $n = \omega$ the question arises as to whether this is legitimate, but since an indirect limit is taken at stage $j^\omega(\kappa)$ the methods of [7] are still applicable.

Therefore, each n-superstrong or hyperstrong cardinal is preserved in *every* generic extension by R, and so any generic extension $V[G]$ preserves all such cardinals. $\dashv$

3.6 1-extendible cardinals and morasses

In the case of forcing the GCH in Chapter 2, there was no major difficulty in proving that 1-extendible cardinals were preserved. In particular, the fact that any embedding j witnessing 1-extendibility lifted presented no great problems, essentially because the forcing affecting the domain of j was small — P_κ was a set in H_{κ^+}. To show that 1-extendible cardinals are preserved while forcing morasses to exist, we will have to work significantly harder, as the forcing of interest, $R_{\kappa+1}$, will be class sized from the point of view of H_{κ^+}.

Theorem 76. *Let V be a model of ZFC. There is class generic extension $V[G]$ of V satisfying the GCH, having mangroves at every regular cardinal in $V[G]$, and such that every 1-extendible cardinal of V is 1-extendible in $V[G]$.*

Proof. We of course apply the two stage iteration $P * R$ of the GCH Partial Order from Chapter 2 followed by the Global Mangrove Partial Order above; recall that whilst taking infinite iterations of class forcings is generally problematic, there is no difficulty with 2-stage iterations.We have already seen in Theorems 32 and 74 that if $G_g * G_m$ is $P * R$-generic over V, then $V[G_g * G_m]$ will satisfy ZFC + GCH and have mangroves at every regular cardinal. It therefore only remains to show that any 1-extendible cardinal κ of V will remain 1-extendible in $V[G_g * G_m]$.

So suppose κ is 1-extendible in V, and let $j : H^V_{\kappa^+} \to H^V_{\lambda^+}$ be an elementary embedding witnessing the 1-extendibility of κ. Let j^* be the lift of j to $H^{V[G_g]}_{\kappa^+}$, as obtained in Theorem 32, so j^* witnesses that κ is 1-extendible in $V[G_g]$. By Theorem 35 we may assume that G_g is such that j^* is elementary from $\langle H^{V[G_g]}_{\kappa^+}, G_g(\kappa)\rangle$ to $\langle H^{V[G_g]}_{\lambda^+}, G_g(\lambda)\rangle$ as $\mathcal{L}_{STG}$-structures, without changing $V[G_g]$.

Now the Global Mangrove Partial Order R may be factorised as $R \cong R_{\kappa+1} * \dot{R}^{\kappa+1}$, with $\dot{R}^{\kappa+1}$ forced to be κ^+-closed. Hence,

$$H^{V[G_g * G_m]}_{\kappa^+} = H^{V[G_g * G_{m,\kappa+1}]}_{\kappa^+},$$

where $G_{m,\kappa+1} = G_m \restriction R_{\kappa+1}$.

To lift our embedding

$$j^* : H^{V[G_g]}_{\kappa^+} \to H^{V[G_g]}_{\lambda^+}$$

to

$$j^{**} : H^{V[G_g * G_m]}_{\kappa^+} \to H^{V[G_g * G_m]}_{\lambda^+},$$

we will first lift it to $H^{V[G_g * G_{m,\kappa}]}_{\kappa^+}$, and then again to $H^{V[G_g * G_{m,\kappa+1}]}_{\kappa^+} = H^{V[G_g * G_m]}_{\kappa^+}$.

Lifting j^* to $j^{*\prime} : H_{\kappa^+}^{V[G_g*G_{m,\kappa}]} \to H_{\lambda^+}^{V[G_g*G_{m,\lambda}]}$ is like the case of forcing the GCH while preserving 1-extendible cardinals (that is, Theorem 32) and at first glance seems to pose no problems. In particular, $H_{\kappa^+}^{V[G_g*G_{m,\kappa}]} = H_{\kappa^+}^{V[G_g]}[G_{m,\kappa}]$, and $j^* \restriction R_\kappa$ is the identity function, so the Lifting Lemma gives us that $j^{*\prime} : H_{\kappa^+}^{V[G_g]}[G_{m,\kappa}] \to H_{\lambda^+}^{V[G_g]}[G_{m,\lambda}]$ is elementary. However, we actually want to demonstrate that $j^{*\prime}$ is elementary from

$$\langle H_{\kappa^+}^{V[G_g]}[G_{m,\kappa}], G_g(\kappa), G_{m,\kappa}\rangle$$

to

$$\langle H_{\lambda^+}^{V[G_g]}[G_{m,\lambda}], G_g(\lambda), G_{m,\lambda}\rangle.$$

The Lifting Lemma will certainly accommodate this if the forcing relation is definable for pre-existing class predicates A (thus dealing with $G_g(\kappa)$; $G_{m,\kappa}$ is not a problem since R_κ is still only set sized from the perspective of $H_{\kappa^+}^{V[G_g]}$). But the definition provided by Property 7 of Theorem 19 relies on our ground model having a definable stratification into sets.

There are (at least) two ways to deal with this. First, we could redefine $p \Vdash^* A(\sigma)$, analogously to the redefinition of $p \Vdash^* G(\sigma)$ in the proof of Lemma 34. Specifically, it is not hard to check that

$$p \Vdash^* A(\sigma) \quad \longleftrightarrow \quad \forall q \leq p \exists r \leq q \exists a \in A(r \Vdash^* \sigma = \check{a}) \tag{7'}$$

is an appropriate definition.

Alternatively, we can give away the punch line of the proof that the next lift goes through, and observe that our ground model $\langle H_{\kappa^+}^{V[G_g]}, G_g(\kappa)\rangle$ *does* have a definable stratification into sets. Specifically, every element of $H_{\kappa^+}^{V[G_g]} = H_{\kappa^+}^{V[G_{g,\kappa}]}$ is coded into the Cohen generic $G_g(\kappa)$, and so $H_{\kappa^+}^{V[G_g]} = L_{\kappa^+}[G_g(\kappa)]$. Hence, the $L_\alpha[G_g(\kappa)]$ hierarchy is a stratification of $H_{\kappa^+}^{V[G_g]}$ into sets, $p \Vdash A(\sigma)$ is definable, and the Lifting Lemma gives that $j^{*\prime}$ is indeed elementary from $\langle H_{\kappa^+}^{V[G_g]}[G_{m,\kappa}], G_g(\kappa), G_{m,\kappa}\rangle$ to $\langle H_{\lambda^+}^{V[G_g]}[G_{m,\lambda}], G_g(\lambda), G_{m,\lambda}\rangle$. That is, we have

$$j^{*\prime} : \langle H_{\kappa^+}^{V[G_g*G_{m,\kappa}]}, G_g(\kappa), G_{m,\kappa}\rangle \to \langle H_{\lambda^+}^{V[G_g*G_{m,\lambda}]}, G_g(\lambda), G_{m,\lambda}\rangle$$

elementary.

We next wish to lift $j^{*\prime}$ to $j^{**} : H_{\kappa^+}^{V[G_g*G_{m,\kappa+1}]} \to H_{\lambda^+}^{V[G_g*G_{m,\lambda+1}]}$. Since $\mathbb{P}_\kappa$ enjoys the κ^+-cc, every $\mathbb{P}_\kappa$-name in $V[G_g * G_{m,\kappa}]$ has an equivalent name in $H_{\kappa^+}^{V[G_g*G_{m,\kappa}]}$, so once again this is the same as lifting to $H_{\kappa^+}^{V[G_g*G_{m,\kappa}]}[G_m(\kappa)]$. The forcing partial order $\mathbb{P}_\kappa$ is class-sized from the point of view of $H_{\kappa^+}^{V[G_g*G_{m,\kappa}]}$, so we need a stratification of $H_{\kappa^+}^{V[G_g*G_{m,\kappa}]}$ into sets in $H_{\kappa^+}^{V[G_g*G_{m,\kappa}]}$ in order to apply Theorem 19 and subsequently the Lifting Lemma. Note on the other hand that the pretameness of $\mathbb{P}_\kappa$ is unproblematic for us: we know (from the V perspective) that forcing with $\mathbb{P}_\kappa$ over $H_{\kappa^+}^{V[G_g * G_{m,\kappa}]}$ will give us a model of ZFC$^-$, $H_{\kappa^+}^{V[G_g * G_{m,\kappa+1}]}$, so since pretameness is implied by ZF$^-$ preservation (Proposition 2.17 of [9]), $\mathbb{P}_\kappa$ is pretame. Alternatively, observe that the κ^+-cc directly implies that $\mathbb{P}_\kappa$ is pretame.

Now we have $H_{\kappa^+}^{V[G_g]} = L_{\kappa^+}[G_g(\kappa)]$, so $H_{\kappa^+}^{V[G_g * G_{m,\kappa}]} = H_{\kappa^+}^{V[G_g]}[G_{m,\kappa}] = L_{\kappa^+}[G_g(\kappa), G_{m,\kappa}]$. Thus, the $L_\alpha[G_g(\kappa), G_{m,\kappa}]$ hierarchy is a stratification of $H_{\kappa^+}^{V[G_g * G_{m,\kappa}]}$ into sets, which is definable in $\langle H_{\kappa^+}^{V[G_g * G_{m,\kappa}]}, G_g(\kappa), G_{m,\kappa}\rangle$. So Theorem 19 shows that the forcing relation is definable, and by the Lifting Lemma, if

$$j^{*\prime} \text{``} G_m(\kappa) \subseteq G_m(\lambda),$$

then $j^{*\prime}$ lifts to an elementary

$$j^{**} : H_{\kappa^+}^{V[G_g * G_{m,\kappa}]}[G_m(\kappa)] \to H_{\lambda^+}^{V[G_g * G_{m,\lambda}]}[G_m(\lambda)]$$

as desired. But now $|G_m(\kappa)| = \kappa^+$ and of course $G_m(\kappa)$ is directed, so since $\mathbb{P}_\lambda$ is λ-directed-closed (Lemma 50), there is a lower bound $p \in \mathbb{P}_\lambda$ for $j^{*\prime} \text{``} G_m(\kappa)$. By Theorem 72, there is a $G'_m(\lambda) \subset H_{\lambda^+}^{V[G_g * G_{m,\lambda}]}[G_m(\lambda)]$ which is $\mathbb{P}_\lambda$-generic over $H_{\lambda^+}^{V[G_g * G_{m,\lambda}]}$, contains the master condition p, and such that

$$H_{\lambda^+}^{V[G_g * G_{m,\lambda}]}[G_m(\lambda)] = H_{\lambda^+}^{V[G_g * G_{m,\lambda}]}[G'_m(\lambda)].$$

Therefore, without changing the model $H_{\lambda^+}^{V[G_g * G_{m,\lambda+1}]}$ we can consider $G'_m(\lambda)$ to be the "G'" of the Lifting Lemma (Lemma 27), and thus obtain an elementary embedding

$$j^{**} : H_{\kappa^+}^{V[G_g * G_{m,\kappa+1}]} \to H_{\lambda^+}^{V[G_g * G_{m,\lambda+1}]}.$$

As noted above, $H_{\kappa^+}^{V[G_g * G_{m,\kappa+1}]} = H_{\kappa^+}^{V[G_g * G_m]}$, and likewise with κ replaced by λ, so j^{**} witnesses that κ is 1-extendible in $V[G_g * G_m]$. ⊣

3.7 Universal morasses

We now consider a notion extending that of a $(\kappa, 1)$-morass, in which a predicate at the top of the morass, built up through the morass, codes every subset of κ. These are of interest to practitioners of pcf theory for use in the construction of scales.

Definition 77. *An* augmented κ-morass *is a $(\kappa, 1)$-morass M, along with a set A_x associated to each element x of $\mathcal{S}^M$, such that*

1. *for every* $x \in \mathcal{S}^M$, $A_x \subseteq o(x)$,
2. *for* $w, x \in \mathcal{S}^M$ *with* $l(w) = l(x)$ *and* $o(w) < o(x)$, $A_w = A_x \cap o(w)$, *and*
3. *for* $x, y \in \mathcal{S}^M$ *with* $x \ltimes^M y$, $A_x = (\pi^M_{xy})^{-1}$“A_y.

For any augmented morass $\langle M, A\rangle$, *we denote by* A_κ *the set* $\bigcup_{\tau<\kappa^+} A_{\langle\kappa,\tau\rangle} \subseteq \kappa^+$. *A* κ-universal morass *is an augmented* κ*-morass* $\langle M, A\rangle$ *such that* $\mathcal{P}(\kappa) \subset L[A_\kappa]$.

It turns out that by modifying the mangrove forcing appropriately, we may force a κ-universal morass. The idea is this: assuming we already have a subset A'_κ of κ^+ such that $\mathcal{P}(\kappa) \subset L[A'_\kappa]$, we may force an augmented morass with A'_κ at the top level. However, there are new subsets of κ added by forcing the universal morass itself. Thus, we also want the morass to be self-coding, giving a subset of κ^+ at the top level which encodes the morass itself.

For those familiar with Jensen's definition of a morass, as in [6] for example, this will seem entirely straightforward: at each node of the morass we can encode the morass below that node, and then the morass maps will naturally preserve this information, cohering at level κ of the morass to give a subset of κ^+ encoding the morass itself. However, in the definition of a morass that we have used (Definition 37, taken from [16]), there is no requirement that the morass maps are this "nice". We are confident that by finding the right balance between Jensen's definition of a morass and Velleman's, this elegant solution might be made to go through. This will have the added benefit that attempts to make our suborder homogeneous will have one problem fewer to overcome: since the augmentation of the morass for self-coding can be thought of as occurring after the forcing takes place, it will not interfere with any desired homogeneity. On the other hand, new ideas will in any case be necessary to make our suborder homogeneous while coding up all of the ground model subsets of κ.

In the meantime, we use a different technique to resolve the matter of self-coding. In essence, we shall force our morass to be self-coding by treating the final code for the morass much like the pre-existing subset A'_κ. To do this, we place the restriction on our conditions that they are only allowed to hit morass coding points at which membership or not is decided by the condition itself. The top level subset of the augmented morass that we force will code the given A'_κ at successor ordinals, and the morass itself will be coded at 0 and limit ordinals, so that there will be no problems with S^p being closed under ordinal successors and predecessors for conditions p. For terminological convenience, we shall consider 0 to be a limit ordinal for the rest of this chapter. Note that the function $\omega\cdot : \alpha \mapsto \omega \cdot \alpha$ is the increasing enumeration of all limit ordinals.

In defining our coding we shall work a little harder than is necessary for our main goal, in order to show how close one can get to homogeneity for our partial order, and where it can nevertheless fail. For this, we wish to avoid coding redundant information. It turns out that mangals will be helpful for this, and moreover the following particular kind.

Definition 78. *Given a morass or morass condition M, a* determined mangal *of M is a mangal β of M such that*

1. *θ_β is a limit ordinal, and*
2. *for all $y \in M$ with $l(y) = \beta$, there is a $\tau < \kappa^+$ such that $y \ltimes \langle \kappa, \tau \rangle$.*

We will need the following basic result stemming from the morass axioms, showing that no "splitting" occurs at limit stages of the tree.

Lemma 79. *Let M be a morass or morass condition, and let $\beta \leq \kappa$ or λ^M respectively. Then for all $y \neq y' \in M$ with $l(y) = l(y') = \beta$ and $o(y) < \theta_\beta$ and $o(y') < \theta_\beta$, there are $x \neq x' \in M$ such that $x \ltimes y$, $x' \ltimes y'$, and x and x' are $\ltimes$-incomparable.*

Proof. Take y and y' with $l(y) = l(y') = \beta$ and $o(y') < o(y)$. By M.5, there is an $x \ltimes y$ and a $\nu < o(x)$ such that $\pi_{xy}(\nu) = o(y')$, whence from M.2 we have that $\langle l(x), \nu \rangle \ltimes y'$. Taking $x' = \langle l(x), \nu \rangle \ltimes y'$, then, we are done. ⊣

Lemma 80. *Suppose β is a determined mangal of M. Then the data θ_β, the tree structure given by $\ltimes$ up to level β and directly from level β to level κ, and the corresponding maps π, are all determined by the part of the morass below level β and the morass maps from such lower nodes to level κ.*

Proof. First we reconstruct θ_β. Let $X \subseteq \kappa^+$ be the set of $\tau \in \kappa^+$ such that there is an $x \in M$ with $l(x) < \beta$ and $x \ltimes \langle \kappa, \tau \rangle$, and define an equivalence relation $\sim$ on X by

$$\sigma \sim \tau \leftrightarrow \forall x \big(l(x) < \beta \rightarrow (x \ltimes \langle \kappa, \sigma \rangle \leftrightarrow x \ltimes \langle \kappa, \tau \rangle) \big).$$

Then by Lemma 79, θ_β is in bijection with the set of $\sim$-equivalence classes, and moreover thanks to M.2, θ_β is the order type of the set of $\sim$-equivalence classes ordered by least element. For any $\gamma < \theta_\beta$ and $\tau \in \kappa^+$, $\langle \beta, \gamma \rangle \ltimes \langle \kappa, \tau \rangle$ if and only if τ is in the γ-th $\sim$-equivalence class of X, and similarly for any $x \in M$ with $l(x) < \beta$, $x \ltimes \langle \beta, \gamma \rangle$ if and only if there is a $\tau < \kappa^+$ in the γ-th equivalence class of X such that $x \ltimes \langle \kappa, \tau \rangle$.

We also claim that the corresponding maps π may be recovered. For $x \ltimes \langle\beta,\gamma\rangle$ this is clear: if $\mu < o(x)$ and $\nu < \gamma$, then $\pi_{x\langle\beta,\gamma\rangle}(\mu) = \nu$ if and only if for $\tau < \kappa^+$ such that $x \ltimes \langle\kappa,\tau\rangle$, $\pi_{x\langle\kappa,\tau\rangle}(\mu)$ is in the ν-th equivalence class of X. The maps $\pi_{\langle\beta,\gamma\rangle\langle\kappa,\tau\rangle}$ for $\langle\beta,\gamma\rangle \ltimes \langle\kappa,\tau\rangle$ may now be deduced from the Commutativity morass axiom: for any $\nu < \gamma$ we have by M.5 that there is an $x \ltimes \langle\beta,\gamma\rangle$ and a $\mu < o(x)$ such that $\pi_{x\langle\beta,\gamma\rangle}(\mu) = \nu$, and then $\pi_{\langle\beta,\gamma\rangle\langle\kappa,\tau\rangle}(\nu)$ must be $\pi_{x\langle\kappa,\tau\rangle}(\mu)$. $\dashv$

Because of this "reconstructibility" property, when we code up the morass we need only flag the determined mangals as such, rather than all of the corresponding details. Thus, when we iteratively construct conditions that are sufficiently closed with respect to hitting the morass coding points, we will be able to add on the necessary new top level without adding condition-specific morass coding point information.

We are now ready to define our encoding for the morass. The first $\kappa \cdot \kappa$ morass coding points will be used to encode the values θ_α for $\alpha < \kappa$. The next $\kappa \cdot \kappa$ will be used to encode the morass relation $\ltimes$ below level κ, and the next κ^4 will be used to encode the maps π_{xy} below level κ. The rest of the morass coding points will be used to encode the relation $x \ltimes y$ and maps π_{xy} for y with level κ.

Definition 81. *Let p be a morass condition, and for all $\gamma \leq \lambda^p + 1$ define*

$$\begin{aligned} ND(\gamma) &= \{\alpha < \gamma \mid \alpha \text{ is not a determined mangal of } p\} \\ \iota(\gamma) &= \sum_{\alpha<\gamma} \theta_\alpha, \qquad \text{and} \\ \delta(\gamma) &= \sum_{\alpha \in ND(\gamma)} \theta_\alpha. \end{aligned}$$

Let $m : \iota(\lambda^p+1) \to \mathcal{S}^p$ $n : \delta(\lambda^p+1) \to \mathcal{S}^p$ be the enumeration in lexicographic order of the nodes of p with level in λ^p+1 and $ND(\lambda^p+1)$ respectively. Let $b : \kappa^+ \times \kappa^+ \to \kappa^+$ be the standard bijection according to the max-lex order on $\kappa^+ \times \kappa^+$. Then the morass code *for p is a partial function*

$$c^p : \{\omega \cdot \alpha \mid \alpha \in \kappa^+\} \to 2$$

such that the following hold.

1. *For all $\alpha \in ND(\lambda^p+1)$ and $\zeta < \kappa$,*

$$c^p(\omega \cdot (\kappa \cdot \alpha + \zeta)) = \begin{cases} 1 & \text{if } \zeta < \theta_\alpha \\ 0 & \text{otherwise.} \end{cases}$$

For every β a determined mangal of p and $\zeta < \kappa$, $c^p(\omega \cdot (\kappa \cdot \beta + \zeta)) = 0$.

2. *For* $\alpha < \iota(\lambda^p)$, *let* μ *be the least determined mangal of* p *greater than* $l(m(\alpha))$ *if such exists, and* $\lambda^p + 1$ *otherwise. Then for all* $\zeta < \iota(\mu)$,

$$c^p(\omega \cdot (\kappa^2 + \kappa \cdot \alpha + \zeta)) = \begin{cases} 1 & \textit{if } m(\alpha) \mathrel{\vdash\!\!<}^p m(\zeta) \\ 0 & \textit{otherwise.} \end{cases}$$

   If μ *is a determined mangal, then for all* $\zeta \geq \iota(\mu)$, $c^p(\omega \cdot (\kappa^2 + \kappa \cdot \alpha + \zeta)) = 0$.

3. *For all* $\alpha < \iota(\lambda^p)$, *all* $\zeta < \iota(\mu)$ *with* μ *as in item 2, and all* $\eta < \kappa$,

$$\begin{aligned} &c^p(\omega \cdot (\kappa^2 \cdot 2 + \kappa^2 \cdot \alpha + \kappa \cdot \zeta + \eta)) \\ &\qquad = \begin{cases} 1 & \textit{if } m(\alpha) \mathrel{\vdash\!\!<}^p m(\zeta) \textit{ and } \eta \in \pi_{m(\alpha)m(\zeta)}``(o(m(\alpha)) + 1) \\ 0 & \textit{otherwise.} \end{cases} \end{aligned}$$

   If μ *is a determined mangal, then for all* $\zeta \geq \iota(\mu)$ *and* $\eta < \kappa$,

$$c^p(\omega \cdot (\kappa^2 \cdot 2 + \kappa^2 \cdot \alpha + \kappa \cdot \zeta + \eta)) = 0.$$

4. *For all* $\tau \in S^p$, $\alpha < \delta(\lambda^p + 1)$, *and* $\nu < \kappa^+$,

$$\begin{aligned} &c^p(\omega \cdot (\kappa^3 + \kappa \cdot b((\tau, \nu)) + \alpha)) \\ &\qquad = \begin{cases} 1 & \textit{if } n(\alpha) \mathrel{\vdash\!\!<}^p \langle \kappa, \tau \rangle \textit{ and } \nu \in \pi_{n(\alpha)\langle \kappa, \tau \rangle}``(o(n(\alpha)) + 1) \\ 0 & \textit{otherwise} \end{cases} \end{aligned}$$

5. *At all other limit ordinals* c^p *is undefined.*

Clearly the entire condition p may be reconstructed from the information provided by c^p. Also, our coding scheme is nice with respect to the mangrove $\leq$ relation:

Lemma 82. *If* p *and* q *are morass conditions with* $p \leq q$ *in the mangrove forcing, then* $c^q \subseteq c^p$.

Proof. If new nodes, tree edges and attendant maps are added to extend q to p, these will be coded at limit ordinals on which c^q was left undefined. If new tree edges from nodes in q to nodes at level κ are added, then the node at level κ must be new (by injectivity of f^p) and again the code is only extended at points where it was previously undefined. $\dashv$

Now we must force in a way that respects c^p and our predicate A'_κ, and moreover such that we will be able to respect c^q for extensions q. The modification to the quagmire forcing necessary to achieve this is given by the following definition. As usual, for any subset X of κ^+ we denote by χ_X the characteristic function of X.

Definition 83. *Let A' be a subset of κ^+ such that $\mathcal{P}(\kappa) \subset L[A']$, and let $\chi_{A'}$ be its characteristic function. Let p be a morass condition. Let $\chi^p_A : \kappa^+ \to \kappa^+$ be the partial function defined by*

$$\chi^p_A(\tau) = \begin{cases} \chi_{A'}(\nu) & \text{if } \tau = \nu + 1 \\ c^p(\tau) & \text{if } c^p \text{ is defined on } \tau \end{cases}$$

(and undefined otherwise), and let A^p_κ denote the subset of $\operatorname{dom}(\chi^p_A)$ *with characteristic function χ^p_A, The condition p is a* universal morass condition for A', *or simply* universal morass condition *when A' is clear from the context, if following properties hold.*

1. *The set $S^p \cap Lim$, where Lim denotes the class of limit ordinals, is a subset of* $\operatorname{dom}(c^p)$.
2. *For all $x \in \mathcal{S}^p$, if τ and ν in S^p are such that $x \ltimes^p \langle\kappa, \tau\rangle$ and $x \ltimes^p \langle\kappa, \nu\rangle$, then $(\pi^p_{x\langle\kappa,\tau\rangle})^{-1}``(A^p_\kappa \cap \tau) = (\pi^p_{x\langle\kappa,\nu\rangle})^{-1}``(A^p_\kappa \cap \nu)$.*
3. *For all $x \in \mathcal{S}^p$ with $l(x) \leq \lambda^p$, there is a $y \in p$ such that $l(y) = \kappa$ and $x \ltimes^p y$.*

Let U denote the set of all universal morass conditions for A'. The Universal Morass Partial Order $\mathbb{U}_\kappa$ for A' *is the suborder of $\mathbb{P}_\kappa$ consisting of all universal morass conditions for A', that is, $\mathbb{U}_\kappa = \langle U, \leq\rangle$ where $\leq$ is defined as for the mangrove forcing.*

As with $\mathbb{P}$, we will generally omit the subscript κ from $\mathbb{U}_\kappa$.

We claim that forcing with $\mathbb{U}$ yields a universal morass. The corresponding set A_κ will be $\bigcup_{p\in G} A^p_\kappa$, and the other sets $A(x)$ for x in the generic morass will "filter down" from this top level predicate: if x, y are nodes in the generic morass with $l(y) = \kappa$, then $A(x) = \pi^{-1}_{xy}``(A_\kappa \cap o(y))$. Requirements 2 and 3 ensure that $A(x)$ will be well-defined for all x in the generic morass, and the coherence properties necessary to have an augmented morass will be automatic.

The verification that forcing with $\mathbb{U}$ still yields a morass is much as in Section 3.2, with a little extra work to check that requirements 1–3 of Definition 83 do not cause problems. We shall now walk through the modified versions of the enunciations in Section 3.2, only describing the necessary modifications to the old proofs. In particular, the fact that straightforward verifications go through will not be mentioned.

Lemma 44′. *There is a universal morass condition q with $S^q = \omega$.*

Proof. The construction is exactly as for Lemma 44. Since $\langle 0, 0\rangle$ must be in q, $c^q(0)$ is defined (and equal to 1). Every node of the q constructed is below a unique node at level κ, and so q satisfies all of the requirements of Definition 83. ⊣

In Subsection 3.2.3 regarding μ-equivalence, Lemmas 46 and 47 go through unchanged, but Lemma 48 now fails in general. However, we do not require that lemma for our present purposes.

Proposition 50′ *The poset $\mathbb{U}$ is κ-directed-closed.*

Proof. As in the proof of Proposition 50, given a directed set Y of conditions, we take the union of the conditions and add a new top level if necessary. The requirement of Definition 83 hold below the new top level because they do for every member of Y, and 2 and 3 hold at the new top level by its definition. ⊣

We need to suitably strengthen the hypotheses of Lemma 51 to make the proof go through in the context of $\mathbb{U}$.

Lemma 51′ *Let p and q be universal morass conditions such that $\lambda^p = \lambda^q$ and $p \overset{\lambda^p}{\sim} q$. Suppose further that there is some $S^0 \subset \kappa^+$ such that both S^p and S^q end-extend S^0, $\min(S^q \smallsetminus S^0) \geq \sup(S^p)$, and $(f^p)^{-1}``A^p_\kappa = (f^q)^{-1}``A^q_\kappa$. Then p and q are compatible in $\mathbb{U}$.*

Proof. The added hypothesis ensures that requirement 2 holds of the condition constructed in the proof of Lemma 51, and the other two requirements are immediate. ⊣

Proposition 52′ *The poset $\mathbb{U}$ is κ^+-cc.*

Proof. At the end of the Δ-system argument for Proposition 52, we may add one extra stage of thinning, to ensure that $(f^p)^{-1}``A^p_\kappa$ is the same for all of the conditions p in the set in question, thus making Lemma 51′ applicable. ⊣

It will be convenient to consider the variants of Lemmas 53 and 54 in reverse order.

Lemma 54′ *For any* $p \neq \mathbf{1} \in \mathbb{U}$ *there is a* $q \leq p$ *in* $\mathbb{U}$ *such that* $S^q = S^p$, $\theta^q_{\lambda^q} = ot(S^q)$, *and* $\lambda^q = \lambda^p + \omega^{ot(S^q)}$.

Proof. It is immediate that the q constructed in the proof of Lemma 54 is a universal morass condition. ⊣

Lemma 53′ *For any* $p \neq \mathbf{1} \in \mathbb{U}$ *and any limit ordinal* $\sigma < \kappa^+$, *there is a* $q \leq p$ *in* $\mathbb{U}$ *such that* $S^q \supseteq ((\sigma + \omega) \smallsetminus \sigma)$.

Proof. The only point of concern is to ensure that $\sigma \in \operatorname{dom}(c^q)$. If $\sigma < \kappa^3$, we may first apply Lemma 54′ repeatedly, "gluing together" with Proposition 50′, to obtain a p' with sufficiently large $\lambda^{p'}$ and $\delta(\lambda^{p'})$ that $c^{p'}$ is defined on σ. The desired extension q may then be constructed starting from p' as in the proof of Lemma 53.

If $\sigma = \omega \cdot (\kappa^3 + \kappa \cdot b((\tau, \nu)) + \alpha)$ for some τ, ν and α, then $\tau \leq \sigma$. We may again make a preliminary extension of p to a p' with $\alpha < \delta(\lambda^{p'} + 1)$. If $\tau = \sigma$, then the q constructed in the proof of Lemma 53 satisfies $\sigma \in \operatorname{dom}(c^q)$, and is thus a universal morass condition. Otherwise, by induction we may extend p' to p'' with $\tau \in S^{p''}$, which implies that $\sigma \in c^{p''}$, and so constructing q from this starting point again makes it a universal morass condition. ⊣

We now have all the necessary ingredients for the following analogue of Theorem 55, which we believe merits its own number.

Theorem 84. *Forcing with* $\mathbb{U}_\kappa$ *yields a* κ*-universal morass.*

Proof. Let G be $\mathbb{U}$-generic over V. The morass part M of the universal morass will simply be the direct limit of the conditions in G, as was the case in our original mangrove forcing. The proof that M is indeed a morass proceeds exactly as for Theorem 55.

As discussed earlier, we set $A_\kappa = \bigcup_{p \in G} A^p_\kappa$, and define $A(x)$ for every $x \in M$ by

$$A(x) = \begin{cases} A_\kappa \cap o(x) & \text{if } l(x) = \kappa \\ \pi_{xy}^{-1}\text{``}A(y) & \text{if } x \mathrel{\vdash\!\!\prec} y \text{ and } l(y) = \kappa. \end{cases}$$

By requirements 2 and 3 this uniquely defines $A(x) \subseteq o(x)$ for every $x \in M$. Requirement 2 for an augmented morass is satisfied thanks to morass axiom M.2 and our definition of $A(x)$, and requirement 3 for an augmented morass is immediate from the Commutativity morass axiom.

Therefore, to prove that $\langle M, A\rangle$ so defined is a κ-universal morass in $V[G]$, it only remains to show that every subset of κ in $V[G]$ lies in $L[A_\kappa]$. We have by assumption that $\mathcal{P}(\kappa)^V \subset L[A']$, and A' is coded into A_κ on successor ordinals. Now since $\mathbb{U}$ has the κ^+ chain condition, every element of $\mathcal{P}(\kappa)^{V[G]}$ has a nice name in $H^V_{\kappa^+}$, and therefore in $L[A_\kappa]$. But now the morass M is entirely encoded into A_κ on limit ordinals, and the generic G may readily be reconstructed from the morass M as the set of all universal morass conditions that are substructures of M. Hence, $G \subset L[A_\kappa]$, so within $L[A_\kappa]$ we can correctly evaluate names, and therefore $\mathcal{P}(\kappa)^{V[G]} \subset L[A_\kappa]$. Overall then, we have that $\langle M, A\rangle$ indeed constitutes a universal morass in $V[G]$. ⊣

We may of course also take a reverse Easton iteration of these partial orders $\mathbb{U}_\kappa$ to force a κ-universal morass to exist at every uncountable regular cardinal κ. For this, we will need to interleave Cohen forcings to obtain the sets A' with the forcing posets $\mathbb{U}_\kappa$.

Definition 85. *The* Global Universal Morass Partial Order W *is the reverse Easton iteration forcing with* $\mathrm{Fn}(\kappa^+, 2, \kappa^+) * \mathbb{U}_\kappa$ *at uncountable regular cardinal stages* κ*, and the trivial forcing elsewhere. In the former case, the* A' *from which the partial order* $\mathbb{U}_\kappa$ *is defined is taken to be the subset of* κ^+ *whose characteristic function is the union of the generic for the preceding* $\mathrm{Fn}(\kappa^+, 2, \kappa^+)$.

Theorem 86. *Forcing with the Global Universal Morass Partial Order yields a universe in which the GCH holds and universal morasses exist at every uncountable regular cardinal.*

Proof. This is a straightforward verification as for Theorem 74. Also note that we only need the GCH to hold below κ^+ for the lemmas leading up to Theorem 84, so we need not even start with a model of the GCH. ⊣

Now we consider large cardinal preservation.

Theorem 87. *Forcing with the Global Universal Morass Partial Order* W *preserves all superstrong cardinals of the ground model. Moreover, given any hyperstrong or* n*-superstrong cardinal* κ *of the ground model for* $n \in \omega + 1$*, the* W*-generic* G *may be chosen so as to preserve the large cardinal strength of* κ*.*

Proof. This is exactly as for Theorem 12 of [8]. ⊣

Finally, we consider briefly the question of the homogeneity, or lack thereof, of $\mathbb{U}$. With our carefully defined coding scheme, we can find conditions that seem to almost be sufficiently interchangeable to define an automorphism of $\mathbb{U}$ as was done in Definition 67 for $\mathbb{P}$. Let $\mathrm{supp}(c)$ denote the subset of $\mathrm{dom}(c)$ on which c is not 0.

Proposition 88. *Let* p *and* q *be universal morass conditions. Then there are universal morass conditions* $p' \le p$ *and* $q' \le q$ *such that* $\lambda^{p'} = \lambda^{q'}$, $\theta^{p'}_{\lambda^{p'}} = \theta^{q'}_{\lambda^{q'}}$, $\mathrm{dom}(c^{p'}) = \mathrm{dom}(c^{q'})$, $\lambda^{p'}$ *is a determined mangal of both* p' *and* q', *and* $\mathrm{supp}(c^{p'}) \subseteq S^{p'} = S^{q'} \supseteq \mathrm{supp}(c^{q'})$.

Proof. Having put in the effort to set up our coding carefully in Definition 81, this is now relatively easy. Note that the fact that $\theta^{p'}_{\lambda^{p'}} = \theta^{q'}_{\lambda^{q'}}$ follows from $S^{p'} = S^{q'}$ and the requirement that every node of a universal morass condition lie below a node at level κ. Now, we may construct sequences p_i and q_i for $i < \omega$ with $p_0 = p$, $q_0 = q$, and for every $i \in \omega$,

$$\mathrm{supp}(c^{p_i}) \cup \mathrm{supp}(c^{q_i}) \subseteq S^{p_{i+1}} \cap S^{q_{i+1}}$$

and

$$\mathrm{dom}(c^{p_i}) \cup \mathrm{dom}(c^{q_i}) \subseteq \mathrm{dom}(c^{p_{i+1}}) \cap \mathrm{dom}(c^{q_{i+1}}).$$

Note that this latter requirement implies that $\lambda^{p_{i+1}} \ge \max(\lambda^{p_i}, \lambda^{q_i}) \le \lambda^{q_{i+1}}$. The construction is by combining Lemmas 53′ and 54′ and Proposition 50′; note that in the proofs of all three of these results, the condition constructed has a determined mangrove as the top level, so in particular parts 2 and 3 of the definition of the morass code (Definition 81) do not pose a problem. Applying Proposition 50′ to obtain lower bounds for the sequences $(p_i)_{i\in\omega}$ and $(q_i)_{i\in\omega}$ gives the desired p' and q' respectively, since the new top level in each case is a determined mangal, requiring no new 1s in the morass code. $\dashv$

Unfortunately, this is still not sufficient to be able to define an automorphism interchanging p' and q'. The difficulty lies in the fact that an extension r of p' may be able to "see" extra information about the structure of p' through the required agreement between $(\pi^r_{xy})^{-1}``A^r_\kappa$ and $(\pi^r_{xz})^{-1}``A^r_\kappa$ whenever $x \ltimes^r y$ and $x \ltimes^r z$ with $l(y) = l(z) = \kappa$. Of course, this may contradict what "happens in" q, thus blocking the existence of an automorphism of $\mathbb{U}$ interchanging p' and q'.

Having hit this stumbling block, we leave the question of the coexistence of universal morasses with large cardinals where it stands after Theorem 87, and move on to a whole other combinatorial principle from L.

Chapter 4

Definable Well-Ordering

We wish to force over V to get some class-generic extension $V[G]$ which has a definable well-order, while preserving both the GCH and various large cardinals.

The idea of our forcing is essentially to add arbitrarily large Cohen sets as in Chapter 2, and as we did in Section 3.6, use the fact that every element of $V[G]$ is ordinal definable from A, a class predicate for the added Cohen sets. The twist here is that we want to make A itself definable in $V[G]$. To achieve this, we code up the choices of 0 or 1 by the generic in terms of whether or not some combinatorial principle holds at various cardinals. Doing this while preserving the GCH puts a heavy constraint on which combinatorial principles can be used for such an encoding; indeed, the GCH itself would otherwise be an ideal principle to use as a coding oracle, as in the work of McAloon [15]. However, the existence of $\diamondsuit^*_\kappa$-sequences also fits the bill nicely, without disturbing the GCH.

Coming at this from the other direction, we have a property suitable to be used as an oracle (existence of $\diamondsuit^*_\kappa$-sequences) and we want to use it to encode a definable well-order of the (extension) universe. Instead of using some complicated iteration with lots of bookkeeping, we may simply "let the generic decide" which way to force at each stage. This technique — having an iteration at each stage of which the generic makes an initial decision that determines the rest of the forcing poset at that stage — is not new in other contexts; see for example Theorems 5.27 and 5.33 of [9] and Section 3 of [10].

4.1 Forcing $\diamondsuit^*_{\kappa^+}$ to hold or fail

Recall the following definition.

Definition 89. *Let λ be a regular cardinal, and let $D = \langle D_\alpha \mid \alpha < \lambda\rangle$ be a* *sequence such that for every* $\alpha < \lambda$, $D_\alpha \subset \mathcal{P}(\alpha)$ *and* $|D_\alpha| \leq |\alpha|$. *Then* D *is said to be a* $\diamondsuit^*_\lambda$-sequence *if for every* $X \subset \lambda$, $\{\alpha \in \lambda \,|\, X \cap \alpha \in D_\alpha\}$ *contains a closed unbounded subset of* λ. *The statement* $\diamondsuit^*_\lambda$ *is the statement that a* $\diamondsuit^*_\lambda$*-sequence exists.*

It turns out that there are known partial orders for forcing $\diamondsuit^*_\lambda$ to hold or fail while preserving the GCH, for each infinite successor cardinal λ. Indeed, in the $\lambda = \omega_1$ case, the two directions are given as exercises in [14] (VII H.18–20 for forcing $\diamondsuit^*_{\omega_1}$ to hold, VIII J.3 for forcing it to fail). For the sake of completeness we give the details (for the general case) here.

To avoid the temptation to abuse notation, we make a definition to recast $\diamondsuit^*_\lambda$-sequences in terms of characteristic functions, giving notationally more convenient objects.

Definition 90. *Let* λ *be a regular cardinal and let* $D = \langle D_\alpha | \alpha < \lambda\rangle$ *be a sequence such that for every* $\alpha < \lambda$, $D_\alpha \subset \mathcal{P}(\alpha)$ *and* $|D_\alpha| \leq |\alpha|$. *We say that* d *is a* listing *of* D *if* d *is a function on* λ *with the property that for each* $\alpha < \lambda$, $d(\alpha)$ *enumerates the characteristic functions of the elements of* D_α *in order type* $|D_\alpha|$. *That is,*

i. for each $\alpha < \lambda$, $d(\alpha)$ *is a a function from* $|D_\alpha|$ *to* ${}^\alpha 2$*; and*

ii. for all $\alpha < \lambda$ *and* $\beta < |D_\alpha|$, *there is some* $S \in D_\alpha$ *such that for all* $\zeta < \alpha$, $d(\alpha)(\beta)(\zeta) = 1$ *if and only if* $\zeta \in S$*; and*

iii. for all $\alpha < \lambda$ *and* $S \in D_\alpha$, *there is a unique* $\beta < |D_\alpha|$ *such that for all* $\zeta < \alpha$, $d(\alpha)(\beta)(\zeta) = 1$ *if and only if* $\zeta \in S$.

Usually we will use listings when D is a $\diamondsuit^*_\lambda$-sequence, but it will also be convenient to have this terminology at our disposal when showing that some D of the right form is *not* a $\diamondsuit^*_\kappa$-sequence.

4.1.1 Forcing $\diamondsuit^*_{\kappa^+}$ to hold

To force $\diamondsuit^*_{\kappa^+}$ to hold, we use a version of Jensen's $\diamondsuit^+_{\kappa^+}$ order, as described in the $\kappa^+ = \omega_1$ case in Exercise VII H.18 of [14]. In doing so, we actually force the stronger statement, $\diamondsuit^+_{\kappa^+}$.

Definition 91. *Let* λ *be a regular cardinal, and let* $D = \langle D_\alpha \,|\, \alpha < \lambda\rangle$ *be a sequence such that for every* $\alpha < \lambda$, $D_\alpha \subset \mathcal{P}(\alpha)$ *and* $|D_\alpha| \leq |\alpha|$. *Then* D *is said to be a* $\diamondsuit^+_\lambda$-sequence *if for every* $X \subset \lambda$, *there is a closed unbounded subset* C *of* λ *such that for all* $\alpha \in C$, $X \cap \alpha \in D_\alpha$ *and* $C \cap \alpha \in D_\alpha$. *The statement* $\diamondsuit^+_\lambda$ *is the statement that a* $\diamondsuit^+_\lambda$*-sequence exists.*

Clearly any $\diamondsuit^+_\lambda$-sequence is a $\diamondsuit^*_\lambda$ sequence, and $\diamondsuit^+_\lambda$ implies $\diamondsuit^*_\lambda$. Because of this, we could equally well consider $\diamondsuit^+_{\kappa^+}$ to be our oracle at κ^+, as we will have $\diamondsuit^*_{\kappa^+} \leftrightarrow \diamondsuit^+_{\kappa^+}$ at all "coding points" after we have performed our forcing iteration to obtain a definable well-order. For the sake of simplicity, we will stick with $\diamondsuit^*_{\kappa^+}$ in the arguments of Section 4.2 and onwards. Also observe that definition 90 may of course be applied to give listings of $\diamondsuit^+_\lambda$-sequences.

To force $\diamondsuit^+_{\kappa^+}$, we do not directly construct a $\diamondsuit^+_{\kappa^+}$-sequence, but rather work indirectly, obtaining another structure which "codes up" a $\diamondsuit^+_{\kappa^+}$-sequence. This coding will be in terms of constructibility; the following standard results will be useful, and their proofs will be a good warm-up for the reader in the techniques we need to use! We use the constructible closure of a set containing two functions ${}^{\kappa^+}2$, but of course the number of such functions used is irrelevant (for finitely many functions in any case). We have chosen to use 2 simply because of the context in which we will apply these lemmas.

Lemma 92. *Let κ be a cardinal, let λ denote ${\kappa^+}^V$, and suppose that for some functions $g, h \in {}^\lambda 2$, $L(\{g,h\}) \vDash \lambda = \kappa^+$. Then for any ordinal $\alpha < \lambda$ there is some ordinal σ strictly between α and λ such that*

$$L_\sigma(\{g\restriction\sigma, h\restriction\sigma\}) \vDash |\alpha| \leq \kappa.$$

Proof. Clearly it suffices to consider $\alpha > \kappa$. Let μ be the least $L(\{g,h\})$-cardinal greater than λ such that $L_\mu(\{g,h\}) \vDash \lambda = \kappa^+$. Let Y be the Skolem hull in $L_\mu(\{g,h\})$ of the set $(\alpha+1) \cup \{g,h\}$, and let M be the Mostowski collapse of Y. Note that by the choice of μ and because $\kappa \subset Y$, we have $Y \cap \lambda \in \mathrm{Ord}$: for each $\beta \in Y \cap \lambda$,

$$Y \vDash \text{there is a surjection } \kappa \to \beta,$$

and if $f : \kappa \to \beta$ is a witness to this statement in Y, then $\beta = f``\kappa \subset Y$. Let us denote $Y \cap \lambda$ by σ_0; note that since $|Y| < \lambda$, $\sigma_0 \neq \lambda$. It now follows that the images of g and h under the Mostowski collapsing function are $g\restriction\sigma_0$ and $h\restriction\sigma_0$ respectively. Thus, since $Y \vDash V = L(\{g,h\})$,

$$M \vDash V = L(\{g\restriction\sigma_0, h\restriction\sigma_0\}),$$

and so for some σ_1, which will be less than λ by cardinality considerations, $M = L_{\sigma_1}(\{g\restriction\sigma_0, h\restriction\sigma_0\})$. Taking $\sigma = \max(\sigma_0, \sigma_1)$ we are done. $\dashv$

Lemma 93. *Let κ be any cardinal, and let g and h be elements of ${}^{\kappa^+}2$ such that $L(\{g,h\})$ correctly computes κ^+. Then either there is some $\sigma < \kappa^+$ such that κ^+ is correctly computed in $L(\{g\restriction\sigma, h\restriction\sigma\})$, or for all $\sigma < \kappa^+$, κ^+ is inaccessible in $L(\{g\restriction\sigma, h\restriction\sigma\})$.*

Proof. To avoid confusion about which universe successor cardinals are taken in, let $\lambda = \kappa^{+V}$. Note first that for any $\sigma < \lambda$ a simple condensation argument in $L(\{g \restriction \sigma, h \restriction \sigma\})$ gives that the GCH holds in $L(\{g \restriction \sigma, h \restriction \sigma\})$ above σ. Thus, if λ is weakly inaccessible in $L(\{g \restriction \sigma, h \restriction \sigma\})$, then λ is also strongly inaccessible in that model.

Since regularity is downwardly absolute, it therefore only remains to show that if there is some σ_0 such that λ is a successor cardinal in $L(\{g \restriction \sigma_0, h \restriction \sigma_0\})$, then there is some σ_1 such that λ is the successor of κ in $L(\{g \restriction \sigma_1, h \restriction \sigma_1\})$. So suppose that σ_0 and α in λ are such that

$$L(\{g \restriction \sigma_0, h \restriction \sigma_0\}) \models \alpha \text{ is a cardinal } \wedge \ \lambda = \alpha^+,$$

that is,

$$L(\{g \restriction \sigma_0, h \restriction \sigma_0\}) \models \alpha \text{ is a cardinal} \wedge \forall \beta < \lambda \, \exists f : \alpha \to \beta (f \text{ is a surjection}).$$

Of course, α must be at least κ, and if $\alpha = \kappa$ we are done, so suppose that $\alpha > \kappa$. By Lemma 92, there is a σ_1 strictly between α and λ such that $L_{\sigma_1}(\{g \restriction \sigma_1, h \restriction \sigma_1\}) \models |\alpha| = \kappa$. But then σ_1 must be greater than σ_0, and we get $L(\{g \restriction \sigma_1, h \restriction \sigma_1\}) \models \lambda = \kappa^+$, as desired. $\dashv$

The structure which will code our $\diamondsuit^+_{\kappa^+}$-sequence will be a kind of tree whose branches code all of ${}^{\kappa^+}2$. Recall the following definition.

Definition 94. *Let λ be a regular cardinal. A* λ-tree *is a tree T of height λ such that for every $\alpha < \lambda$, $|Lev_\alpha(T)| < \lambda$. A* λ-Kurepa tree *is a λ-tree with at least λ^+ cofinal branches.*

Also note that when we say that T is a subtree of ${}^{<\kappa^+}2$, we mean that $T \subseteq {}^{<\kappa^+}2$, and T is closed in ${}^{<\kappa^+}2$ under initial segments.

The following key lemma is actually just one direction of an equivalence, given in the ω_1 case as part of exercise VI.9 of [14]. However, the proof of the other direction is surprisingly involved, and is not necessary for our purposes, so we omit it. The interested reader is directed to Corollary II 7.11 of [14] for the main component of that argument.

Lemma 95. *Let κ^+ be a successor cardinal. If there is a κ^+-tree T which is a subtree of ${}^{<\kappa^+}2$, and an $h \in {}^{\kappa^+}2$ such that*

$$\forall f \in {}^{\kappa^+}2 \, \exists g \in {}^{\kappa^+}2 (f \in L(\{g, h\}) \wedge g \textit{ is the union of a cofinal branch in } T)$$

then $\diamondsuit^+_{\kappa^+}$ holds.

Proof. Let T and h be as in the statement of the lemma. Since $|T| = \kappa^+$, T may be coded into a single element of ${}^{\kappa^+}2$. We may therefore assume without loss of generality that $T \in L(h)$. Similarly, we may assume that h codes up a surjection from κ to α for each $\alpha < \kappa^+$, so that $L(h)$ correctly computes κ^+.

For each $\alpha < \kappa^+$ and $g' \in T \cap {}^{\alpha}2$, let $q(\alpha, g')$ denote the least ordinal $\rho > \alpha$ such that

$$L_\rho(\{g', h \restriction \alpha\}) \prec L_{\kappa^+}(\{g', h \restriction \alpha\}),$$

and let $q(\alpha) = \sup\{q(\alpha, g') \mid g' \in T \cap {}^{\alpha}2\}$. We define our claimed $\diamondsuit^+_{\kappa^+}$-sequence D by

$$D_\alpha = \mathcal{P}(\alpha) \cap L_{q(\alpha)}(\{T \cap {}^{\alpha}2, h \restriction \alpha\})$$

for each $\alpha < \kappa^+$.

To show that D is a $\diamondsuit^+_{\kappa^+}$-sequence, suppose that A is any subset of κ^+. Let $f_A : \kappa^+ \to 2$ be the characteristic function of A, and let $g_A \in {}^{\kappa^+}2$ be the union of a cofinal branch in T such that $f_A \in L(\{g_A, h\})$, as given by the hypothesis of this lemma. Let us define σ_A to be the least ordinal σ such that $\kappa \le \sigma < \kappa^+$ and $L(\{g_A \restriction \sigma, h \restriction \sigma\})$ correctly computes κ^+, if such a σ exists, and let $\sigma_A = \kappa$ otherwise.

Consider the model $L_{\kappa^{++}}(\{g_A, h\})$, where the subscript κ^{++} here denotes κ^{++} as computed in $L(\{g_A, h\})$. Observe that since $A \in \mathcal{P}(\kappa^+) \cap L(\{g_A, h\})$, a condensation argument gives that $A \in L_{\kappa^{++}}(\{g_A, h\})$. Now, it is easy to define a well-order on the transitive closure of $\{g_A, h\}$, from which we obtain a definable well-order on $L_{\kappa^{++}}(\{g_A, h\})$ in the usual way, and hence definable (formula by formula, and relative to $\mathrm{trcl}(\{g_A, h\})$) Skolem functions. For any $X \subseteq L_{\kappa^{++}}(\{g_A, h\})$, let $\mathrm{cl}_A(X)$ denote the closure of X in $L_{\kappa^{++}}(\{g_A, h\})$ with respect to these Skolem functions.

Let

$$C_A = \{\alpha < \kappa^+ \mid \alpha \ge \sigma_A \wedge \alpha = \mathrm{cl}_A(\alpha \cup \{A, g_A, h\}) \cap \kappa^+\}.$$

Clearly C_A is closed unbounded in κ^+. Let α be an element of C_A; we will show that $A \cap \alpha \in D_\alpha$ and $C_A \cap \alpha \in D_\alpha$.

Let $Y_{A,\alpha}$ denote $\mathrm{cl}_A(\alpha \cup \{A, g_A, h\})$, and let $M_{A,\alpha}$ denote the Mostowski collapse of $Y_{A,\alpha}$. Since

$$L_{\kappa^{++}}(\{g_A, h\}) \models V = L(\{g_A, h\})$$

we have

$$Y_{A,\alpha} \models V = L(\{g_A, h\})$$

and hence, letting $\mathrm{Mc}_{A,\alpha}$ denote the Mostowski collapsing function $Y_{A,\alpha} \to M_{A,\alpha}$, we get

$$M_{A,\alpha} \models V = L(\{\mathrm{Mc}_{A,\alpha}(g_A), \mathrm{Mc}_{A,\alpha}(h)\}).$$

But now $Y_{A,\alpha} \cap \kappa^+ = \alpha$ by assumption on α, so $\mathrm{Mc}_{A,\alpha}(\kappa^+) = \alpha$, and $\mathrm{Mc}_{A,\alpha}(A) = A \cap \alpha$, $\mathrm{Mc}_{A,\alpha}(g_A) = g_A \restriction \alpha$ and $\mathrm{Mc}_{A,\alpha}(h) = h \restriction \alpha$. Thus,

$$M_{A,\alpha} \vDash V = L(\{g_A \restriction \alpha, h \restriction \alpha\}),$$

and since $\alpha = \mathrm{Mc}_{A,\alpha}(\kappa^+) \in M_{A,\alpha}$ and $|M_{A,\alpha}| < \kappa^+$, we obtain that for some δ with $\alpha < \delta < \kappa^+$, $M_{A,\alpha} = L_\delta(\{g_A \restriction \alpha, h \restriction \alpha\})$.

We now claim that $q(\alpha) > \delta$. Applying Lemma 93, and recalling that $\alpha \geq \sigma_A$, there are two cases to consider.

Case I: $L(\{g_A \restriction \alpha, h \restriction \alpha\})$ correctly computes κ^+. Then a standard condensation argument in $L(\{g_A \restriction \alpha, h \restriction \alpha\})$ gives that

$$L_{\kappa^+}(\{g_A \restriction \alpha, h \restriction \alpha\}) \vDash |\alpha| = \kappa.$$

Of course since g_A is a cofinal branch in T, $g_A \restriction \alpha \in T \cap {}^\alpha 2$, and so

$$L_{q(\alpha, g_A \restriction \alpha)}(\{g_A \restriction \alpha, h \restriction \alpha\}) \vDash |\alpha| = \kappa$$

whence

$$L_{q(\alpha)}(\{g_A \restriction \alpha, h \restriction \alpha\}) \vDash |\alpha| = \kappa.$$

But we have already seen that

$$L_\delta(\{g_A \restriction \alpha, h \restriction \alpha\}) \vDash \alpha = \kappa^+,$$

so $q(\alpha)$ must be greater than δ.

Case II: κ^+ is inaccessible in $L(\{g_A \restriction \alpha, h \restriction \alpha\})$. Then $L_{q(\alpha, g\restriction\alpha)}(\{g \restriction \alpha, h \restriction \alpha\})$ contains cardinals of $L(\{g \restriction \alpha, h \restriction \alpha\})$ greater than α. On the other hand, $L\delta(\{g \restriction \alpha, h \restriction \alpha\}) \vDash \forall \beta > \alpha(|\beta| = \alpha)$, so in particular $L\delta(\{g \restriction \alpha, h \restriction \alpha\})$ cannot contain any cardinals of $L(\{g \restriction \alpha, h \restriction \alpha\})$. Thus, $q(\alpha) > \delta$ in this case also.

Now

$$\begin{aligned} A \restriction \alpha \in \mathcal{P}(\alpha) \cap L_\delta(\{g_A \restriction \alpha, h \restriction \alpha\}) &\subseteq \mathcal{P}(\alpha) \cap L_{q(\alpha)}(\{g_A \restriction \alpha, h \restriction \alpha\}) \\ &\subseteq \mathcal{P}(\alpha) \cap L_{q(\alpha)}(\{T \cap {}^\alpha 2, h \restriction \alpha\}) \\ &= D_\alpha \end{aligned}$$

so D is indeed a $\diamondsuit^*_{\kappa^+}$-sequence.

To see that D is moreover a $\diamondsuit^+_{\kappa^+}$-sequence, observe that

$$\begin{aligned} C_A \cap \alpha &= \{\beta < \alpha \mid \beta \geq \sigma_A \wedge \beta = \mathrm{cl}_A(\beta \cup \{A, g, h\}) \cap \kappa^+\} \\ &= \{\beta < \alpha \mid \beta \geq \sigma_A \wedge \mathrm{Mc}_{A,\alpha}(\beta) = \mathrm{Mc}_{A,\alpha}(\mathrm{cl}_A(\beta \cup \{A, g, h\}) \cap \kappa^+)\} \\ &= \{\beta < \alpha \mid \beta \geq \sigma_A \wedge \beta = \mathrm{Mc}_{A,\alpha}(\mathrm{cl}_A(\beta \cup \{A, g, h\})) \cap \alpha\} \end{aligned}$$

Since our Skolem functions are definable relative to $\mathrm{trcl}\{g_A, h\}$, the respective Mostowski collapses of these functions, considered as classes of $Y_{A,\alpha}$, will be the classes of $L_\delta(\{g_A\restriction\alpha, h\restriction\alpha\})$ defined by the same formulae. Thus, letting cl^δ_A denote the Skolem closure in $L_\delta(\{g_A \restriction \alpha, h \restriction \alpha\})$ with respect to these canonical Skolem functions, we have

$$\begin{aligned} C_A \cap \alpha &= \{\beta < \alpha \mid \beta \geq \sigma_A \wedge \beta = \mathrm{cl}^\delta_A(\mathrm{Mc}_{A,\alpha}(\beta \cup \{A, g, h\})) \cap \alpha\} \\ &= \{\beta < \alpha \mid \beta \geq \sigma_A \wedge \beta = \mathrm{cl}^\delta_A(\beta \cup \{A \cap \alpha, g\restriction\alpha, h\restriction\alpha\}) \cap \alpha\} \end{aligned}$$

Using a satisfaction predicate on $L_\delta(\{g_A\restriction\alpha, h\restriction\alpha\})$, we see that this set is in $L_{\delta+2}(\{g_A\restriction\alpha, h\restriction\alpha\}) \subset L_{q(\alpha)}(\{g_A\restriction\alpha, h\restriction\alpha\})$, and so it lies in D_α Hence, D is a $\diamondsuit^+_{\kappa^+}$ sequence. ⊣

Exercises VII H.18–20 of [14] show how to force $\diamondsuit^+_{\omega_1}$ by forcing an ω_1-Kurepa tree to exist and observing that it satisfies the extra hypotheses of Lemma 95. We shall follow the method given there in doing the same for κ^+, but we shall mildly simplify the forcing used to obtain a κ^+-Kurepa tree. Our modified forcing also has the benefit of being κ^+-closed rather than merely κ^+-distributive, which will be convenient when iterating the forcing.

Definition 96. *For any cardinal κ, let P_{κ^+} denote (for the remainder of this chapter) the partial order whose elements are pairs $\langle X, \alpha\rangle$, where $X \subset {}^{\kappa^+}2$, $\alpha < \kappa^+$, and $|X| \leq \kappa$. For $\langle X, \alpha\rangle, \langle Y, \beta\rangle \in P_{\kappa^+}$, say that $\langle Y, \beta\rangle \leq \langle X, \alpha\rangle$ if and only if $Y \supseteq X$, $\beta \geq \alpha$, and for all $f \in Y$ there is a $g \in X$ such that $f\restriction\alpha = g\restriction\alpha$.*

For any condition $\langle X, \alpha\rangle \in P_{\kappa^+}$, let

$$T_{\langle X,\alpha\rangle} = \{f\restriction\beta \in {}^{\leq\alpha}2 \mid f \in X \wedge \beta \leq \alpha\}.$$

Intuitively, the condition $\langle X, \alpha\rangle$ can be thought of as determining that the initial segment up to level α of the ultimate κ^+-Kurepa tree will be $T_{\langle X,\alpha\rangle}$, and further, forcing that every element of X will be the union of a cofinal branch of the κ^+-Kurepa tree.

Lemma 97. *For any cardinal κ, P_{κ^+} is κ^+-closed.*

Proof. Since κ^+ is regular, if we have a descending sequence $\langle\langle X_\gamma, \alpha_\gamma\rangle \mid \gamma < \delta\rangle$ in P_{κ^+} for some $\delta < \kappa^+$, it is clear that $\langle\bigcup_{\gamma<\delta} X_\gamma, \sup_{\gamma<\delta}(\alpha_\geq)\rangle$ will be a lower bound for the sequence in P^+_κ. ⊣

Lemma 98. *For any successor cardinal κ^+, P_{κ^+} has the $(2^\kappa)^+$-cc.*

Proof. This follows from the fact that conditions $\langle X, \alpha\rangle$ with the same α and the same set $\{f \restriction \alpha \mid f \in X\}$ are always compatible. Observe that since $|X| \leq \kappa$ and $|\alpha| \leq \kappa$, each such set can be represented by a surjection from κ to α along with κ-many subsets of κ, so there are 2^κ-many distinct such sets. $\dashv$

Proposition 99. *Suppose that* $M \models ZFC + GCH$ *and* κ^+ *is a successor cardinal of* M*. Then if* G *is* P_{κ^+}*-generic over* M*,* $M[G] \models ZFC + GCH + \diamondsuit^*_{\kappa^+}$*.*

Proof. We have seen that under the GCH, P_{κ^+} is κ^+-closed and enjoys the κ^{++}-cc. Therefore cardinals are preserved. Moreover, under the GCH we have $|P_{\kappa^+}| = \kappa^{++}$, so from the κ^{++}-cc, there are at most $(\kappa^{++})^{(\kappa^+)} = \kappa^{++}$ antichains in P_{κ^+}. Thus, a nice names argument gives that $(2^\lambda)^{M[G]} = (2^\lambda)^M$ for all $\lambda \geq \kappa^+$, and with κ^+-closure as well we have that the GCH is preserved.

In $M[G]$, let

$$X^G = \bigcup\{X \mid \exists\alpha(\langle X, \alpha\rangle \in G)\}$$

and let

$$T^G = \bigcup\{T_{\langle X,\alpha\rangle} \mid \langle X\alpha\rangle \in G\}.$$

For any given $\gamma < \kappa^+$, it is clearly dense in P_{κ^+} to have $\alpha > \gamma$. Since $|\mathrm{Lev}_\beta(T_{\langle X,\alpha\rangle})| \leq |X| \leq \kappa$ for each $\langle X, \alpha\rangle \in P_{\kappa^+}$ and $\beta \leq \alpha$, it follows that T^G is a κ^+-tree.

Clearly every element of X^G is the union of a cofinal branch of T^G. Now, given any $f \in (2^{(\kappa^+)})^M$, it is dense to have some $g \in X$ such that $g(\xi) = f(\xi)$ for all $\xi < \kappa^+$ sufficiently large. Thus, for each $f \in (2^{(\kappa^+)})^M$, there is some $\langle X, \alpha\rangle \in G$ and g with the same tail as f such that $g \in X$. It follows that $|X^G| \geq 2^{(\kappa^+)^M} = (\kappa^+ +)^{M[G]}$, so T^G is in fact a κ^+-Kurepa tree.

Now, let $h \in {}^{\kappa^+}2$ be such that T^G and $\mathcal{P}(\kappa)$ are in $L(h)$; this is possible for T^G because it is a κ^+-tree, and for $\mathcal{P}(\kappa)$ because the GCH holds. We claim that T^G and h satisfy the hypotheses of Lemma 95, thus yielding a $\diamondsuit^+_{\kappa^+}$-sequence, and in particular certifying that $\diamondsuit^*_{\kappa^+}$ holds.

First observe that for any $f \in ({}^{\kappa^+}2)^M$, there is a $g \in X^G$ such that $f \in L(\{g, h\})$: as noted above, there is some $g \in G$ and $\gamma < \kappa^+$ such that for all $\xi \geq \gamma$, $f(\xi) = g(\xi)$. But now $f \restriction \gamma \in L(\{g, h\})$ since $\mathcal{P}(\kappa)$ is, and so $f \in L(\{g, h\})$.

Now, suppose that $f \in ({}^{\kappa^+}2)^{M[G]}$. Then there is a nice name $\dot{f}$ for f of the form

$$\dot{f} = \bigcup_{\beta < \kappa^+} (\{\langle\beta, 0\rangle\} \times A_{\beta,0}) \cup (\{\langle\beta, 1\rangle\} \times A_{\beta,1}),$$

where for each $\beta < \kappa^+$ and $i \in 2$, $A_{\beta,i}$ is an antichain of P_{κ^+}. Since P_{κ^+} satisfies the κ^{++}-cc, each A_β has cardinality at most κ^+. Hence, since each element of P_{κ^+} may be coded by a single element of ${}^{\kappa^+}2$, $\dot{f}$ itself may be coded by a single element f' of ${}^{\kappa^+}2$. That is, $\dot{f} \in L(f')$.

Since $f' \in ({}^{\kappa^+}2)^M$, there is a $g \in X$ such that $f' \in L(\{g, h\})$, whence $\dot{f} \in L(\{g, h\})$. If we could show that $G \in L(\{g, h\})$, it would follow that $f \in L(\{g, h\})$, and we would be done. However, G has size κ^{++}, so we cannot in general expect to have coded it up in g and h. On the other hand, h was chosen specifically so that $T^G \in L(h)$, and we claim that T^G will be sufficient to correctly evaluate $\dot{f}$.

In $M[G]$, T^G in fact determines G: take the set of conditions

$$\bar{G} = \big\{\langle X, \alpha\rangle \;\big|\; \big(T_{\langle X,\alpha\rangle} = T^G \cap {}^{\leq\alpha}2\big) \wedge \big(\forall\beta < \kappa^+(T_{\langle X,\beta\rangle} \subseteq T^G \cap {}^{\leq\beta}2)\big)\big\}.$$

Clearly $G \subseteq \bar{G}$; but also each element of $\bar{G}$ is compatible with every element of G, so by genericity $\bar{G} \subseteq G$.

But now this means that the predicate after the "|" in the definition of $\bar{G}$, which in particular is definable in T^G alone, determines whether or not a given condition $\langle X, \alpha\rangle$ is in G. Therefore $\dot{f}$ may be correctly evaluated in $L(\{g, h\})$ as claimed, and our f indeed lies in $L(\{g, h\})$. ⊣

4.1.2 Forcing $\diamondsuit^*_{\kappa^+}$ to fail

Making $\diamondsuit^*_{\kappa^+}$ fail will simply be a matter of adding κ^{++}-many Cohen subsets of κ^+ and observing that $\diamondsuit^*_{\kappa^+}$ will not hold in the extension. Recall the $\mathrm{Fn}(I, J, \lambda)$ notation of [14], given in Definition 1; our forcing to make $\diamondsuit^*_{\kappa^+}$ fail will be $\mathrm{Fn}(\kappa^{++}, 2, \kappa^+)$. The arguments in this subsection actually work for any uncountable regular cardinal λ, so we present them at this level of generality, although for later sections we will only need the case when $\lambda = \kappa^+$ for some κ.

The following observation is of course standard.

Lemma 100. *Suppose $M \models$ ZFC + GCH, and λ is a regular cardinal of M. Then forcing with* $\mathrm{Fn}(\lambda^+, 2, \lambda)$ *over M preserves cardinals and the GCH.*

Proof. The partial order $\mathrm{Fn}(\lambda^+, 2, \lambda)$ is clearly λ closed, and enjoys the λ^+-cc (lemma VII 6.10 of [14] with the GCH), so all cardinals are preserved. The cardinality of $\mathrm{Fn}(\lambda^+, 2, \lambda)$ is $(\lambda^+)^{<\lambda} \times 2^{<\lambda} = \lambda^+$, so there are at most $(\lambda^+)^\lambda = \lambda^+$ antichains in $\mathrm{Fn}(\lambda^+, 2, \lambda)$. Thus, a nice names argument shows that there are at most $((\lambda^+)^\mu)^M = \mu^+$ subsets of μ in the generic extension for any $\mu \geq \lambda$, so with λ-closure it follows that the GCH is maintained. ⊣

The verification that $\mathrm{Fn}(\lambda^+, 2, \lambda)$ destroys $\diamondsuit^*_\lambda$ is presented in [5] for the case when $\lambda = \omega_1$, but with appropriate modifications the argument can be lifted to work for any regular uncountable λ. We present this modified argument here.

Proposition 101. *Suppose that* $M \models$ *ZFC+ GCH and* λ *is a regular cardinal of* M. *Then if* G *is* $\mathrm{Fn}(\lambda^+, 2, \lambda)$*-generic over* M, $M[G] \models$ *ZFC+ GCH+*$\neg\diamondsuit^*_\lambda$.

Proof. We first claim that forcing with $\mathrm{Fn}(\lambda^+, 2, \lambda)$ destroys any $\diamondsuit^*_\lambda$-sequence of M. Let D be a $\diamondsuit^*_\lambda$-sequence of M. For notational convenience, we may assume by expanding the sets D_α if necessary that for every $\alpha < \lambda$, $|D_\alpha| = |\alpha|$. Let d be a listing of D; the assumption of the last sentence thus becomes the statement that for each $\alpha < \lambda$, $d(\alpha)$ has domain $|\alpha|$.

Let $\dot{C}$ be a name for a club subset of λ in $M[G]$, and let $\dot{F}$ name $\bigcup G$. We claim that in $M[G]$, the subset of λ with characteristic function $\bigcup G$ is not correctly guessed by D on all elements of $\dot{C}^G$. Suppose to the contrary that there is some $p \in G$ such that

$$p \Vdash (\dot{C} \text{ is a club in } \check{\lambda}) \wedge \forall \alpha \in \dot{C} \exists \delta < |\alpha| (\dot{F} \restriction \alpha = \check{d}(\alpha)(\delta)).$$

So that we may smoothly deal with the successor and inaccessible cases simultaneously, let $\gamma = \kappa + 1$ for κ such that $\kappa^+ = \lambda$ if λ is a successor cardinal, and let $\gamma = \lambda$ otherwise. By induction on rank in ${}^{<\gamma}2$, we may construct conditions $\langle p_s \mid s \in {}^{<\gamma}2\rangle$ and ordinals $\langle \alpha_s \mid s \in {}^{<\gamma}2\rangle$ such that the following properties hold.

i. $p_\emptyset \leq p$ and $p_\emptyset \Vdash \check{\alpha}_\emptyset \in \dot{C}$.

ii. For every $\beta < \lambda$, $(\mathrm{dom}(p_\beta) \cap \lambda) \in \lambda$.

iii. $s \subseteq t$ implies $p_t \leq p_s$ and $\alpha_t \geq \alpha_s$

iv. For $i \in 2$ we have $\mathrm{dom}(p_{s^\frown\langle i\rangle}) \ni (\mathrm{dom}(p_s) \cap \lambda)$, and

$$p_{s^\frown\langle i\rangle}(\mathrm{dom}(p_s) \cap \lambda) = i.$$

v. For $i \in 2$ we have $\alpha_{s^\frown\langle i\rangle} > \mathrm{dom}(p_s) \cap \lambda$ and $p_{s^\frown\langle i\rangle} \Vdash \check{\alpha}_{s^\frown\langle i\rangle} \in \dot{C}$.

vi. If $t \in {}^{\beta+1}2$ then $\mathrm{dom}(p_t) \cap \lambda \geq \sup(\{\alpha_s \mid s \in {}^{\beta}2\})$.

vii. $\mathrm{dom}(s) = \mathrm{dom}(t)$ implies $\mathrm{dom}(p_s) \cap \lambda = \mathrm{dom}(p_t) \cap \lambda$.

viii. If $s \in {}^{\eta}2$ for η is a limit ordinal, then $p_s = \bigcup_{\beta<\eta} p_{s\restriction\beta}$ and $\alpha_s = \sup(\{\alpha_{s\restriction\beta} \mid \beta < \eta\})$.

Indeed, we may construct such $\langle p_s \mid s \in {}^{<\lambda}2\rangle$ and $\langle \alpha_s \mid s \in {}^{<\lambda}2\rangle$ by first extending p as appropriate for (i) and (ii), at successor stages extending to satisfy (iv), (v), (vi) and (vii) in that order while respecting (ii) and (iii), and at limit stages satisfying (viii).

We claim that for all $s \in {}^{<\lambda}2$, $p_s \Vdash \check{\alpha}_s \in \dot{C}$. Of course from the definitions we need only check this for s with domain a limit ordinal. But for such s, $p_s \Vdash \check{\alpha}_{s\restriction\beta} \in \dot{C}$ for all $\beta < \mathrm{dom}(s)$, and so since $p \geq p_s$ forces that $\dot{C}$ is a club and $\alpha_s = \sup(\{\alpha_{s\restriction\beta} \mid \beta < \gamma\})$, $p_s \Vdash \check{\alpha}_s \in \dot{C}$.

Note that for any limit ordinal $\gamma < \lambda$, α_s for $s \in {}^{\gamma}2$ is independent of the choice of s: for t with domain less than γ,

$$\alpha_{t^\frown\langle i,j\rangle} > \mathrm{dom}(p_{t^\frown\langle i\rangle}) \cap \lambda \geq \sup(\{\alpha_s \mid s \in {}^{\mathrm{dom}(t)}2\}),$$

so

$$\alpha_s = \sup(\{\alpha_{s\restriction\beta} \mid \beta < \gamma\}) = \sup(\{\alpha_t \mid \mathrm{dom}(t) < \gamma\}).$$

Hence, let us denote α_s for $s \in {}^{\gamma}2$ by α_γ. Observe further that because we have terms $\mathrm{dom}(p_t) \cap \lambda$ interleaving with terms α_t in the above inequalities, and $p_s = \bigcup_{\beta<\gamma} p_{s\restriction\beta}$ for $s \in {}^{\gamma}2$, we have $\mathrm{dom}(p_s) \cap \lambda = \alpha_\gamma$ for such s.

But now let μ be the least cardinal such that $|\alpha_\mu| = \mu$; such a $\mu < \lambda$ can easily be found by a typical closure argument. For each $s \in {}^{\mu}2$ we have a condition p_s such that $p_s \restriction \lambda \in {}^{\alpha_\mu}2$. Moreover, (iv) dictates that for $s \neq t \in {}^{\alpha}2$, $p_s \restriction \lambda \neq p_t \restriction \lambda$. Thus, we have 2^μ distinct elements of ${}^{\alpha_\mu}2$, so not all of them can be of the form $d(\alpha_\mu)(\delta)$ for $\delta < |\alpha_\mu| = \mu$. So let $s \in {}^{\mu}2$ be such that for all $\delta < \mu$, $p_s \restriction \lambda = p_s \restriction \alpha_\mu \neq d(\alpha_\mu)(\delta)$. But then

$$p_s \Vdash (\check{\alpha}_\mu \in \dot{C}) \wedge \forall\delta < |\check{\alpha}_\mu|(\dot{F} \restriction \alpha_\mu = (p_s \check{\restriction} \alpha_\mu) \neq \check{d}(\check{\alpha}_\mu)(\delta))$$

contradicting the fact that $p_s \leq p$. We have therefore shown that our forcing indeed destroys any ground model $\diamondsuit^*_\lambda$-sequences.

Now suppose that some D of the right form to be a $\diamondsuit^*_\lambda$-sequence (that is, satisfying the assumptions of Definition 90) is added by $\mathrm{Fn}(\lambda^+, 2, \lambda)$; we wish to show that D is not in fact a $\diamondsuit^*_\lambda$-sequence in $M[G]$. Let $d \in M[G]$ be a listing of D. Since $\mathrm{Fn}(\lambda^+, 2, \lambda)$ is λ-closed and hence adds no new $< \lambda$-tuples of ground model sets, $d(\alpha)$ is an element of M for each $\alpha < \lambda$. Therefore, d can be named by a name $\dot{d}$ which involves for each $\alpha < \lambda$ a single antichain of $\mathrm{Fn}(\lambda^+, 2, \lambda)$ to determine $d(\alpha)$. To be precise, if A_α is a maximal antichain of conditions that determine $d(\alpha)$, and for $p \in A_\alpha$ we write f_p for that function such that $p \Vdash \dot{d}(\check{\alpha}) = \check{f}_p$, then we may take $\dot{d}$ to be

$$\dot{d} = \bigcup_{\alpha<\lambda} \Big\{ \Big\langle \big\{\langle\check{\alpha}, 1\rangle\big\}, 1 \Big\rangle, \Big\langle \big(\big\{\langle\check{\alpha}, 1\rangle\big\} \cup \big\{\langle\check{f}_p, p\rangle \mid p \in A_\alpha\big\}\big), 1 \Big\rangle \Big\}.$$

Since $\mathrm{Fn}(\lambda^+, 2, \lambda)$ has the λ^+-chain condition, $|\bigcup_{\alpha<\lambda} A_\alpha| \leq \lambda$, and so

$$\Big| \bigcup_{\alpha<\lambda} \bigcup_{p\in A_\alpha} \mathrm{dom}(p) \Big| \leq \lambda.$$

Thus, there is some common upper bound $\gamma < \lambda$ on the domains of those conditions p appearing in $\dot{d}$. Now

$$\mathrm{Fn}(\lambda^+, 2, \lambda) \cong \mathrm{Fn}(\gamma, 2, \lambda) \times \mathrm{Fn}([\gamma, \lambda^+), 2, \lambda),$$

and if $G_\gamma = G \cap \mathrm{Fn}(\gamma, 2, \lambda)$, we have $d \in M[G_\gamma]$. Of course,

$$\mathrm{Fn}([\gamma, \lambda^+), 2, \lambda) \cong \mathrm{Fn}(\lambda^+, 2, \lambda),$$

and since $\mathrm{Fn}(\gamma, 2, \lambda)$ is λ-closed,

$$\mathrm{Fn}(\lambda^+, 2, \lambda)^{M[G_\gamma]} = \mathrm{Fn}(\lambda^+, 2, \lambda)^M.$$

So by what we have already shown, if $G^\gamma = G \cap \mathrm{Fn}([\gamma, \lambda^+), 2, \lambda)$, then d does not represent a $\diamondsuit^*_\lambda$-sequence in $M[G_\gamma][G^\gamma] = M[G]$. Therefore, there are no $\diamondsuit^*_\lambda$-sequences in $M[G]$. $\dashv$

4.2 Forcing a definable well-order

In this section we exhibit our forcing which yields a universe with a definable well order. There is much flexibility in the definition we shall present, a fact which we will later exploit when trying to preserve various different kinds of large cardinals.

As discussed at the start of the chapter, the general idea of our forcing is to use $\diamondsuit^*_\kappa$ at various κ to act as an oracle, coding up a proper class of ordinals from which our well-order will be defined. In fact, we further obtain that the extension $V[G]$ is of the form $L[A]$ for A a definable class in $V[G]$. In some sense this is the closest to L we can hope to get while trying to preserve very large cardinals — it follows from Kunen's theorem that $V \neq L(x)$ for any set x if V contains strong cardinals, and of course A cannot be taken to be definable over L as that would give $L[A] = L$. On the other hand, it is possible to have properties very different from those of L coded into A (for example, the failure of $\diamondsuit^*_{\kappa^+}$ for many cardinals κ!), so in itself this should not be thought of as a resolution of the outer model programme.

We retain the notation from Subsection 4.1.1 of P_{κ^+} being our forcing to produce a $\diamondsuit^*_{\kappa^+}$-sequence. For ease of notation let us set $Q_{\kappa^+} = \mathrm{Fn}(\kappa^{++}, 2, \kappa^+)$, the forcing that quashes all $\diamondsuit^*_{\kappa^+}$-sequences. We assume that $V \models \mathrm{GCH}$, which may of course be forced as in Chapter 2.

We wish to force at various successor cardinals to "switch $\diamondsuit^*$ on or off", and then use this as an oracle to make the universe well-orderable. Perhaps the most natural sequence of cardinals at which to do this would be simply the class of *all* infinite successor cardinals. However, for consideration of large cardinal preservation, it will be convenient to use more restricted classes of successor cardinals. Thus, let us denote by c a function from ordinals to successor cardinals which is the increasing enumeration of the cardinals at which we wish to "code". For present purposes, one may think of c being the function $\aleph_{\cdot+1} : \alpha \mapsto \aleph_{\alpha+1}$, enumerating all infinite successor cardinals, but the results will be stated in full generality for later applicability. Of course there will be concerns regarding the absoluteness of the class enumerated by c, and these shall be addressed as they arise; unless otherwise stated, $c(\alpha)$ should be taken to be computed in the ground model V.

For each ordinal α, let $R_{c(\alpha)}$ be the partial order given by combining disjoint copies of $P_{c(\alpha)}$ and $Q_{c(\alpha)}$ below a new maximum element in the obvious way. For concreteness, let us set $1_{R_{c(\alpha)}} = \emptyset$, and let

$$R_{c(\alpha)} = \{1_{R_{c(\alpha)}}\} \cup (\{0\} \times P_{c(\alpha)}) \cup (\{1\} \times Q_{c(\alpha)}).$$

For $r_0, r_1 \in R_{c(\alpha)}$, $r_1 \leq r_0$ if and only if either $r_0 = 1_{R_{c(\alpha)}}$, or $r_0 = \langle i, r_0' \rangle$ and $r_1 = \langle i, r_1' \rangle$ for some $i \in 2$ and $r_1', r_0' \in P_{c(\alpha)} \cup Q_{c(\alpha)}$ such that $r_1' \leq_{P_{c(\alpha)}} r_0'$ or $r_1' \leq_{Q_{c(\alpha)}} r_0'$. Clearly $R_{c(\alpha)}$ will have cardinality $c(\alpha)^+$, be $c(\alpha)$-closed, and have the $c(\alpha)^+$-cc, since these statements are true of both $P_{c(\alpha)}$ and $Q_{c(\alpha)}$. For $\gamma \in \mathrm{Ord}$ not of the form $c(\alpha)$ for some α, let R_γ be the trivial forcing.

Definition 102. *The* $\diamondsuit^*$ Oracle Partial Order S *is the reverse Easton iteration of* $\dot{R}_\alpha$ *as above for* $\alpha \in Ord$.

Note that with only trivial forcings used between cardinals, Easton support is the same as taking direct limits at inaccessibles and inverse limits elsewhere.

Lemma 103. *If* $V \models ZFC + GCH$ *and* G *is generic for the* $\diamondsuit^*$ *Oracle Partial Order* S *over* V, *then* $V[G]$ *satisfies* $ZFC + GCH$ *and has the same cardinals as* V.

Proof. The arguments are as in the relevant part of Theorem 74. The tameness of S comes directly from Proposition 26. To prove that cardinals and the GCH are preserved, we argue by induction on the length of the iteration. Successor stages are immediate from the fact that R_{κ^+} is κ^+-closed and κ^{++}-cc. For limit stages λ, cardinals and the GCH are preserved below λ by the closure of the tail parts of the iteration S_λ. There is a dense suborder of S_λ of size at most λ^+ if λ is singular or λ is λ is regular, preserving the cardinals and GCH above λ^+, or λ in the regular case. It therefore only remains to show that in the λ singular case, the GCH holds at λ and λ^+ is preserved, and this follows by considering nice names for subsets of λ built up as the union of nice names for subsets of smaller cardinals. $\dashv$

In particular, note that the class of successor cardinals is unchanged at each stage of the iteration, so if $c = \aleph_{\cdot+1}$, then c is absolute.

Considering the factorisation of S as $S_\kappa * S^\kappa$, with S^κ being κ-closed, also gives the following.

Lemma 104. *Forcing with S preserves inaccessible cardinals.* $\dashv$

Another basic property of this forcing we shall need is that after applying it, $\diamondsuit^*$ holds at exactly those points in the sequence c where we expect it to: those $c(\alpha)$ such that $\langle 0, 1_{P_{c(\alpha)}}\rangle \in G(c(\alpha))$.

Lemma 105. *Let $V \models$ ZFC + GCH and let G be S-generic over V. Then for every $\alpha \in$ Ord, $V[G] \models \diamondsuit^*_{c(\alpha)}$ if and only if $\langle 0, 1_{P_{c(\alpha)}}\rangle \in G(c(\alpha))$.*

Proof. Let κ^+ be of the form $c(\alpha)$, and consider the factorisation of S as $S_{\kappa^+} * R_{\kappa^+} * S^{\kappa^+ +1}$. Clearly $V[G_{\kappa^+} * G(\kappa^+)] \models \diamondsuit^*_{\kappa^+}$ if and only if $\langle 0, 1_{P_{c(\alpha)}}\rangle \in G(c(\alpha))$, and $S^{\kappa^+ +1}$ is κ^{++}-closed, so any $\diamondsuit^*_{\kappa^+}$-sequence of $V[G_{\kappa^+} * G(\kappa^+)]$ remains a $\diamondsuit^*_{\kappa^+}$-sequence of $V[G]$, and no new $\diamondsuit^*_{\kappa^+}$-sequences are added by $G^{\kappa^+ +1}$. $\dashv$

Theorem 106. *Let $V \models$ ZFC + GCH, and let S be the $\diamondsuit^*$ Oracle Partial Order as defined above, with a definable "coding points" function c absolute for extension by S. If G is S-generic over V, then there is a definable class of ordinals A of $V[G]$ such that $V[G] = L[A]$. In particular, $V[G] = \mathit{HOD}^{V[G]}$, and there is a definable well-order on $V[G]$.*

Proof. The class A will of course be $\{\alpha \in \mathrm{Ord} \mid \diamondsuit^*_{c(\alpha)} \text{ holds}\}$. Clearly A is definable in $V[G]$.

Consider an arbitrary $x \in V[G]$, let $\kappa = |\mathrm{trcl}(\{x\})|$, and let $f : \mathrm{trcl}(\{x\}) \to \kappa$ be a bijection. Let $b_0, b_1 : \kappa \to \kappa$ be the two components of a definable bijection $b = (b_0, b_1) : \kappa \to \kappa \times \kappa$, and let $X = \{\alpha \in \kappa \mid f^{-1} \circ b_0(\alpha) \in f^{-1} \circ b_1(\alpha)\}$. Then the Mostowski collapse of $\langle \kappa \times \kappa, b``X\rangle$ is $\mathrm{trcl}(\{x\})$, and so to show that $x \in L[A]$ it will suffice to show that $X \in L[A]$. Since X is a subset of κ, we must have $X \in V[G_{\kappa+1}]$, since $S^{\kappa+1}$ is κ^+-closed.

We claim that X must appear in the choices of the tail part of the generic. To make this precise, consider the set

$$\begin{aligned} D_X \;=\; \big\{ s \in S^{\kappa+1} \mid \exists \gamma \in \mathrm{Ord}\, \forall \alpha < \kappa \\ \big(\alpha \in X \to \Vdash_{S^{\kappa+1}} s(c(\gamma+\alpha)) \leq_{\dot{R}_{c(\gamma+\alpha)}} \langle 0, \dot{1}_{P_{c(\gamma+\alpha)}}\rangle \wedge \\ \alpha \notin X \to \Vdash_{S^{\kappa+1}} s(c(\gamma+\alpha)) \leq_{\dot{R}_{c(\gamma+\alpha)}} \langle 1, \dot{1}_{Q_{c(\gamma+\alpha)}}\rangle \big) \big\}. \end{aligned}$$

It is clearly dense in $S^{\kappa+1}$: for any condition $p \in S^{\kappa+1}$, simply take γ such that $c(\gamma)$ contains the support of p, and then extend p to $s \in D_X$ according to X. Since we are only extending in κ-many coordinates, there will be no concerns about the "right kinds" of limit being taken — the right kind will always be an inverse limit.

Now, by Lemma 105, if $s \in D_X \cap G$ with associated ordinal γ as in the definition of D_X, then

$$X = \{\alpha < \kappa \mid \diamondsuit^*_{c(\gamma+\alpha)} \text{ holds}\}$$

and so certainly lies in $L[A]$ — X is the subsequence of A of length κ starting at γ. Overall, we therefore have that indeed, $V[G] = L[A]$, from which $V[G] = \mathrm{HOD}^{V[G]}$ and the existence of a definable well order of $V[G]$ are immediate. $\dashv$

4.3 Preserving large cardinals

One could follow the basic scheme of the previous chapters in trying to lift the embeddings witnessing the large cardinal strength of various cardinals. However, in many cases one will quickly run into the problem which we worked so hard in Chapter 3 to overcome for the morass forcing: there will be master conditions that need to lie in the generic, giving problems if there are many large cardinals to be preserved. Unlike in the morasses case, our partial order is inherently inhomogeneous, and so a new approach is required.

The solution to this problem comes from the extra flexibility we have because we are interested in forcing a global principle, rather than a local principle at every regular cardinal. We can "thin out" our forcing partial order, still obtaining a definable well-order of the extension universe, but finessing the issue of master conditions by making the forcing trivial at every point where master conditions might be required.

To facilitate this thinning out, we make the following definitions.

Definition 107. *Suppose that φ is a formula in one variable, and more specifically, that:*

1. $\varphi(\kappa) \equiv$ *"κ is measurable", or*
2. *for some ordinal η, $\varphi(\kappa) \equiv$ "κ is η-strong", or*
3. *for some $n \in \omega + 1$, $\varphi(\kappa) \equiv$ "κ is n-superstrong", or*
4. $\varphi(\kappa) \equiv$ *"κ is hyperstrong", or*
5. *for some definable function g, $\varphi(\kappa) \equiv$ "κ is $g(\kappa)$-supercompact", or*

6. *for some ordinal η, $\varphi(\kappa) \equiv$ "κ is η-extendible", or*

7. *for some $m \in \omega$, $\varphi(\kappa) \equiv$ "κ is m-huge".*

*A cardinal λ is a φ-*bound *if λ is an infinite cardinal, λ is not measurable, and if $\varphi(\kappa)$ holds for any $\kappa < \lambda$, then there is an elementary embedding j with critical point κ witnessing the fact that $\varphi(\kappa)$ holds, such that*

- *i. $\varphi(\kappa) \equiv$ "κ is measurable"$\to \kappa < \lambda$ (that is, no extra requirement),*
- *ii. $\varphi(\kappa) \equiv$ "κ is η-strong" $\to \beth^+_{\kappa+\eta} < \lambda$,*
- *iii. $\varphi(\kappa) \equiv$ "κ is n-superstrong" $\to \beth_{j^n(\kappa)} < \lambda$,*
- *iv. $\varphi(\kappa) \equiv$ "κ is hyperstrong" $\to \beth_{j(\kappa)+1} < \lambda$,*
- *v. $\varphi(\kappa) \equiv$ "κ is $g(\kappa)$-supercompact" $\to g(\kappa)^{<\kappa} < \lambda$,*
- *vi. $\varphi(\kappa) \equiv$ "κ is η-extendible" $\to \zeta < \lambda$ for ζ such that $j : V_{\kappa+\eta} \to V_\zeta$, and*
- *vii. $\varphi(\kappa) \equiv$ "κ is m-huge" $\to 2^{j^m(\kappa)} < \lambda$.*

We say that λ is a minimal φ-bound *if for every cardinal $\nu < \lambda$ there is a cardinal κ which is not a φ-bound such that $\nu \leq \kappa < \lambda$.*

Clearly one may construct minimal φ-bounds by the usual methods for obtaining fixed points.

Unless otherwise specified, φ shall henceforth denote one of the listed large cardinal properties 1–7, and for convenience we shall refer to cardinals κ satisfying $\varphi(\kappa)$ as *φ-cardinals*. This list of large cardinal properties, which will be the ones that are preserved in Theorem 110, should by no means be thought of as encompassing all large cardinals for which the techniques of this chapter are applicable. Rather, it is a representative list of well-known large cardinals each witnessed by boundedly many elementary embeddings (indeed, 1!), so that bounds may be constructed for them. For reasons that will become apparent, it would also be of interest (and moreover straightforward) to include large cardinals of the form "φ a limit of φ" (for example, a measurable limit of measurables). However, if we do not wish to move to a more general statement, we must draw the line somewhere!

Once we have a φ-bound, the succeeding cardinals will remain φ-bounds for some time. The following lemma in this direction will be sufficient for our purposes.

Lemma 108. *Suppose $\aleph_\beta$ is a φ-bound. Then for all γ less than the least inaccessible greater than $\aleph_\beta$ (or all γ if no such inaccessible exists), $\aleph_{\beta+\gamma}$ is a φ-bound.*

Proof. This is immediate from the fact that the least cardinal κ which is *not* a φ-bound above a given φ-bound will satisfy $\varphi(\kappa)$, and hence be inaccessible. $\dashv$

With this fact, we are ready to define the points at which we shall perform our coding.

Definition 109. *A cardinal λ is a* φ-coding point *if*

1. *λ is a successor cardinal, and*
2. *λ is a φ-bound, and*
3. *if there is a cardinal $\kappa > \lambda$ such that $\varphi(\kappa)$, there is a minimal φ-bound $\mu \leq \lambda$ such that every inaccessible cardinal less than or equal to λ is less than or equal to μ.*

Thus, coding points come after each minimal φ-bound, going on until the next inaccessible cardinal, or indefinitely if there is no next φ-cardinal.

Theorem 110. *Let $V \models ZFC + GCH$, and let φ be one of the large cardinal properties 1–7 listed in Definition 107. Let $c : Ord \to Ord$ be the enumeration of the φ-coding points of V, and let S be the $\diamondsuit^*$ Oracle Partial Order defined from c. Suppose G is S-generic over V. Then there is $V[G]$-definable class of ordinals A such that $V[G] = L[A]$. Further, if κ is a φ-cardinal in V that is not a limit of φ-cardinals, then in $V[G]$ remains a φ-cardinal.*

So for example, we can preserve all measurable cardinals that are not limits of measurables; see below for a discussion of extensions strengthening this.

Proof. Let us denote the class of all φ-coding points by C and the class of all minimal φ-bounds by B. Note that if the universe contains any φ-cardinals, B will not be closed unbounded in Ord, but it will enjoy countable closure, which will be crucial to the argument. The class A as in the statement of the theorem will again be $\{\alpha \in \mathrm{Ord} \mid \diamondsuit^*_{c(\alpha)^{V[G]}} \text{ holds}\}$, but now we are faced with the problem that B, and hence C and c, may not be absolute from V to $V[G]$. Even though we shall show that φ-cardinals that are not limits of φ-cardinals are preserved, there are two problematic possibilities to consider: that there may be new such cardinals introduced, removing elements of B^V from $B^{V[G]}$, and that there may be new embeddings witnessing the large cardinal strength of large cardinals from V, potentially adding new φ-bounds below old minimal φ-bounds and thus changing elements of B.

However, these issues will not cause problems for us. Since $V[G]$ satisfies the Axiom of Replacement with respect to formulas involving a predicate for

V, the usual argument to show that countably closed unbounded sets have countably closed unbounded intersection goes through for $B^V \cap B^{V[G]}$, and we see that there are unboundedly many cardinals in $B^V \cap B^{V[G]}$. We claim that this agreement is sufficient to ensure that every element of $V[G]$ is coded up by the class A of φ-coding points of $V[G]$ at which $\diamondsuit^*$ holds.

So suppose $x \in V[G]$, and let X be a subset of $\kappa = |\mathrm{trcl}(\{x\})|$ coding up x, as in the proof of Theorem 106. We must strengthen the observation that X appears in the choices in the tail of the generic as follows: for every β such that $c(\beta) \geq \kappa$, X appears in the choices in the generic between $c(\beta)$ and the least inaccessible greater than $c(\beta)$ (if one exists), where in each case $c(\beta)$ is to be computed in V. To see this, we modify D_X: letting $\iota_{c(\beta)}$ denote the least inaccessible greater than $c(\beta)$ if such exists or Ord otherwise, let

$$\begin{aligned} D_{X,\beta} \quad = \quad & \big\{ s \in S^{\kappa+1} \,|\, \exists \gamma \in \iota_{c(\beta)} \,\big(\gamma \geq \beta \wedge \forall \alpha < \kappa \\ & \qquad \big(\alpha \in X \to \Vdash_{S^{\kappa+1}} s(c(\gamma+\alpha)) \leq_{\dot{R}_{c(\gamma+\alpha)}} \langle 0, \dot{1}_{P_{c(\gamma+\alpha)}} \rangle \wedge \\ & \qquad\ \alpha \notin X \to \Vdash_{S^{\kappa+1}} s(c(\gamma+\alpha)) \leq_{\dot{R}_{c(\gamma+\alpha)}} \langle 1, \dot{1}_{Q_{c(\gamma+\alpha)}} \rangle \,\big)\big)\big\}. \end{aligned}$$

Because a direct limit is taken at $\iota_{c(\beta)}$, we have for any $s \in S^{\kappa+1}$ that $\mathrm{supp}(s) \cap \iota_{c(\beta)}$ is bounded in $\iota_{c(\beta)}$, so we may extend $s \restriction \iota_{c(\beta)}$ to an element of $D_{X,\beta} \cap S^{[\kappa+1,\iota_{c(\beta)})}$, and then "re-attach the tail of s" to get an extension of s in $D_{X,\beta}$. Hence, for each β with $c(\beta) \geq \kappa$, the class $D_{X,\beta}$ is dense in $S^{\kappa+1}$, and so has non-empty intersection with $G^{\kappa+1}$.

Now because inaccessibles are absolute between V and $V[G]$ (Lemma 104), if $\lambda \in B^V \cap B^{V[G]}$ is greater than κ and ι_λ is the least inaccessible greater than λ, then $[\lambda, \iota_\lambda) \cap \mathrm{Succ} \subset C^V \cap C^{V[G]}$, where Succ denotes the class of successor cardinals. Therefore, taking β to be least such that $c(\beta) \geq \lambda$, $s \in D_{X,\beta} \cap G^{\kappa+1}$, and γ witnessing that $s \in D_{X,\beta}$, we have that $c^V``[\gamma, \gamma+\kappa) = c^{V[G]}``[\zeta, \zeta+\kappa)$ for some ordinal ζ, and so indeed, $X \in L[A]$. Hence, we have shown that $V[G] = L[A]$.

It now remains to show that for any φ-cardinal κ of V that is not a limit of φ-cardinals remains a φ-cardinal in $V[G]$. In each case the respective representation lemma for M (Lemma 10, 12, 15 or 17) will be crucial.

Measurable Cardinals. Let $j : V \to M$ be an ultrapower embedding witnessing the measurability of κ with $j(\kappa)$ least. We shall construct an S^M-generic G^* over M in $V[G]$, such that we can lift j to $j^* : V[G] \to M[G^*]$. Note that the φ-coding points of V less than κ are in fact bounded below κ since the class of measurable cardinals is. Hence, S^M is S^V up to stage κ, and is trivial from κ to $j(\kappa)$. We may therefore take $G^*_\kappa = G_\kappa$, trivially extend to $G^*_{j(\kappa)}$, and have a lift of j to $j' : V[G_\kappa] \to M[G^*_{j(\kappa)}]$. To define $G^{*j(\kappa)}$, note that every element of M has the form $j(f)(\kappa)$, where $f : \kappa \to V$ is a function in V, and so every element of $M[G^*_{j(\kappa)}]$ has the form $\sigma_{G_{j(\kappa)}}$, where σ has the form $j(f)(\kappa)$. We claim that the filter on $S^{j(\kappa)}$ generated by $j'``G^{j(\kappa)}$ is $(S^{j(\kappa)})^M$-generic over M.

So suppose that D is a dense class in $M[G^*_{j(\kappa)}]$, defined (in $M[G^*_{j(\kappa)}]$) relative to the parameter $d \in M[G^*_{j(\kappa)}]$ by $D = \{x \mid \psi(x, d)\}$. Let σ be an $S^M_{j(\kappa)}$-name in M such that $d = \sigma_{G^*_{j(\kappa)}}$, and let $f : \kappa \to V$ in V be such that $\sigma = j(f)(\kappa)$. Since κ is not a measurable-coding point, $(S^\kappa)^V$ is κ^+-closed, and we see that it is dense for $s \in (S^\kappa)^V$ to extend an element of the class $D_\alpha = \{x \mid \psi(x, f(\alpha)_{G_\kappa})\}$ of V whenever $\alpha \in \kappa$ with $f(\alpha)$ an S_κ-name and D_α dense in $(S^\kappa)^V$. Therefore, we may take such an s lying in G^κ. By elementarity, it follows that $j(s)$ extends an element of D. Hence, the filter generated by $j'``G^\kappa$ is indeed $(S^{j(\kappa)})^{M[G^*_{j(\kappa)}]}$-generic over M. By the Lifting Lemma, it follows that there is an elementary embedding $j^* : V[G] \to M[G^*]$ lifting j, and so κ is measurable in $V[G]$.

η-Strong Cardinals. We may assume that our η-strong embedding $j : V \to M$ is an extender ultrapower embedding, with every element of M having the form $j(f)(a)$, with a a finite tuple from $|V_{\kappa+\eta}|^+$ and f a function from $[\kappa]^{|a|}$ to V in V. As in the measurable cardinal case, we show that the filter generated by $j'``G^\kappa$ is $(S^{j(\kappa)})^{M[G^*_{j(\kappa)}]}$, observing that by the definition of a φ-bound, S^κ is trivial up to at least stage $|V_{\kappa+\eta}|^{++}$, and so we have the requisite closure to make the argument go through. We can therefore lift j to $j^* : V[G] \to M[G^*]$. We may also conclude that $V^{V[G]}_{\kappa+\eta} \subseteq M[G^*]$ from a nice names argument, since $V^V_{\kappa+\eta} \subseteq M$, $G_{\kappa+\eta} = G^*_{\kappa+\eta}$, and $S_{\kappa+\eta}$ is trivial beyond some bound below κ. Hence, κ is η-strong in $V[G]$.

n-Superstrong Cardinals and Hyperstrong Cardinals. The argument is again the same, this time with our extender models having elements of the form $j(f)(a)$ with a in $V_{j^n(\kappa)}$ and f with domain $V_{j^{n-1}(\kappa)}$ in the case of n-superstrong cardinals, and $a \in V_{j(\kappa)+1}$ and f with domain $V_{\kappa+1}$ in the case of hyperstrong cardinals. The required level of agreement between $V[G]$ and $M[G^*]$ again follows from a nice names argument, noting in the hyperstrong case that $V_{j(\kappa)+1} \in M \leftrightarrow H_{j(\kappa)^+} \in M$.

$g(\kappa)$-Supercompact Cardinals. In this case we may take the elements of M to be of the form $j(f)(j``g(\kappa))$, so in $V[G_\kappa]$ we consider all $x \in \mathcal{P}_\kappa(g(\kappa))$; again, because of our definition of a $g(\kappa)$-supercompact coding point, the argument goes through without difficulty. To show that $M[G^*]$ is closed under taking $g(\kappa)$-tuples, note that for any $g(\kappa)$-tuple from $M[G^*]$ in $V[G]$, we may consider an $g(\kappa)$-tuple t of names for its elements in $V[G]$, where the names are in M. All $g(\kappa)$-tuples in $V[G]$ of elements of V are in $V[G_{g(\kappa)}]$ by closedness of the tail of the iteration, and so since $S_{g(\kappa)}$ is trivial beyond some bound below κ, there is a nice name $\dot{t}$ for t with only $g(\kappa)$ elements. Therefore, since M is closed with respect to taking $g(\kappa)$-tuples in V, $\dot{t} \in M$, and since G^*_κ is the same as G_κ, $t \in M[G^*_{j(\kappa)}]$. But then the original $g(\kappa)$-tuple of elements from $M[G^*]$ is in $M[G^*]$, as desired.

η-Extendible Cardinals. On the domain and range of j, the nontrivial part of S is bounded below κ, so this is trivial.

m-Huge Cardinals. This is much like the $g(\kappa)$-supercompact case. The elements of M may be taken to be of the form $j(f)(j``(j^m(\kappa)))$ where the domain of f is $\mathcal{P}(j^m(\kappa))$, so the assumed $2^{j^m(\kappa)}$-closure of S^κ is what we need to construct the M-generic for the lifting. Closure of $M[G^*]$ with respect to taking $j^m(\kappa)$-tuples is exactly as in the $g(\kappa)$-supercompact case.

This completes the verification. $\dashv$

One may wonder if the restriction on which φ-cardinals are preserved (that is, only those that are not limits of φ-cardinals) can be lifted. However, some kind of restriction like this is necessary for our technique. We are using the fact that the set of coding points is bounded below every cardinal that we lift. Hence, by Fodor's theorem, we cannot hope to lift all φ-cardinals from a universe where they form a stationary set in Ord.

On the other hand, we can always mollify this problem, by restricting it to a smaller and smaller class of large cardinals. By thinning out the class of coding point while still keeping it unbounded in Ord, the above arguments will still go through at all of the cardinals that were previously preserved, but with new cardinals added to the list of large cardinals that stay large. For example, if there are boundedly many measurable limits of measurables, and we thin out the measurable-coding points to only use those "directly after" a measurable limit of measurables, until there are no more, then we will still preserve all measurable cardinals that are not limits of measurables, but we will also preserve those measurable limits of measurables that are not limits of measurable limits of measurables. If there is a proper class of measurable limits of measurables, the "thinned out" class of coding points is even easier to describe: it is simply the set of φ-coding points for $\varphi \equiv$"κ is a measurable limit of measurables". Indeed, this can be done not just for φ limits of φ-cardinals, but for any proper class sequence of cardinals at whose limits we don't mind preservation failing. So for example, we may deduce the following.

Theorem 111. *Suppose there is a proper class of ω-superstrong cardinals, and let δ be an arbitrary ordinal. Then a definable well-order of the universe may be forced while preserving all measurable, η-strong for $\eta < \delta$, n-superstrong for $n \in \omega + 1$, hyperstrong, η-supercompact for $\eta < \delta$, η-extendible for $\eta < \delta$, and m-huge for $m \in \omega$ cardinals that are not limits of ω-superstrong cardinals.*

Proof. We take φ to be the (size $|\delta|$) disjunction of all of the stated large cardinal properties for the sake of defining φ-bounds, but for defining the coding points we only consider the minimal φ-bounds that are minimal above an ω-superstrong cardinal (and as before, take the block of coding points starting at such minimal φ-bounds to have length the next inaccessible). The arguments from Theorem 110 for each individual case will all go through unaffected, as long as the cardinal in question is not a limit of ω-superstrong cardinals. $\dashv$

Of course we have chosen to use a very strong cardinal in this theorem for the simple reason that generally, stronger large cardinals *are* limits of weaker large cardinals. The reader who finds the assumption of a proper class of ω-superstrong cardinals unpalatably strong can of course substitute in its place a proper class of whatever they wish, and the theorem will remain true, albeit vacuous in some cases.

Bibliography

[1] James E. Baumgartner. Iterated forcing. In *Surveys in Set Theory*, number 87 in London Mathematical Society Lecture Note Series, pages 1–59. Cambridge University Press, 1983.

[2] J. Burgess. On a set-mapping problem of Hajnal and Máté. *Acta Scientiarum Mathematicarum*, 41:283–288, 1979.

[3] Douglas Burke. Generic embeddings and the failure of box. *Proceedings of the American Mathematical Society*, 123(9):2867–2871, 1995.

[4] James Cummings and Ernest Schimmerling. Indexed squares. *Israel Journal of Mathematics*, 131:61–99, 2002.

[5] Keith J. Devlin. Variations on $\diamondsuit$. *Journal of Symbolic Logic*, 44(1):51–58, 1979.

[6] Keith J. Devlin. *Constructibility.* Springer, Berlin, 1984.

[7] Natasha Dobrinen and Sy D. Friedman. Homogeneous iteration and measure one covering relative to HOD. To appear.

[8] Sy D. Friedman. Large cardinals and L-like universes. To appear in *Set theory: recent trends and applications, Quaderni di Matematica.*

[9] Sy D. Friedman. *Fine Structure and Class Forcing.* Number 3 in de Gruyter Series in Logic and Its Applications. de Gruyter, Berlin, 2000.

[10] Joel David Hamkins. The lottery preparation. *Annals of Pure and Applied Logic*, 101(2–3):103–146, 2000.

[11] Thomas Jech. *Set Theory.* Springer, Third Millenium edition, 2003.

[12] Akihiro Kanamori. Morasses in combinatorial set theory. In A.R.D. Mathias, editor, *Surveys in Set Theory*, pages 167–196. Cambridge University Press, 1983.

[13] Akihiro Kanamori. *The Higher Infinite*. Springer, 2nd edition, 2003.

[14] Kenneth Kunen. *Set Theory*. North-Holland, 1980.

[15] Kenneth McAloon. Consistency results about ordinal definability. *Annals of Mathematical Logic*, 2(4):449–467, 1970/71.

[16] Daniel J. Velleman. Morasses, diamond, and forcing. *Annals of Mathematical Logic*, 23(2-3):199–281, 1982.

[17] Daniel J. Velleman. Simplified morasses. *The Journal of Symbolic Logic*, 49(1):257–271, 1984.

Index

- A_κ, 61
- P_{κ^+} ($\diamondsuit^*_{\kappa^+}$ forcing), 77
- R_κ (iterand of $\diamondsuit^*$ Oracle Forcing), 83
- S ($\diamondsuit^*$ Oracle forcing), 83
- S^p (morass condition level κ), 26
- $\mathbb{P}^\alpha$ (mangrove forcing above λ^α), 51
- $\mathbb{P}_\kappa$ (mangrove forcing), 28
- $\mathcal{L}_{STG}$, 19
- $\mathcal{S}$
  - of a morass, 24
  - of a morass condition, 26
- $\diamondsuit^+_\lambda$, 72
  - -sequence, 72
- $\diamondsuit^*$ Oracle Partial Order, 83
- $\diamondsuit^*_\lambda$, 72
  - -sequence, 72
- $\leq$ relation
  - $\diamondsuit^*$ Oracle Forcing iterand, 83
  - $\diamondsuit^*_{\kappa^+}$ forcing, 77
  - mangrove forcing, 27
- $\overset{\mu}{\sim}$, 30
- φ-bound, 86
  - minimal, 86
- φ-cardinal, 86
- q^p_r, 52
- 1-extendible, 2

- almost κ-indiscernible, 44
- Axiom I2, 2

- bamboo, 29

- closed, 32
  - directed-, 32
- coding point, φ-, 87
- complete embedding, 47

- determined mangal, 62
- directed, 32
- directed-closed, κ-, 32

- equivalent
  - $\leq p$ (for tameness), 9
  - μ- (morass conditions), 30
- extender, 2
  - ultrapower, 2
- extendible
  - η-, 4
  - 1-, 2

- Factor Lemma, 12
- field, 3

- GCH, 15
- GCH Partial Order, 16
- Global Mangrove Partial Order, 56
- Global Universal Morass Partial Order, 68

- huge, m-, 5
- hyperstrong, 2

- inner model programme, iii

- Knaster, 38
- Kurepa tree
  - λ-, 74

- Left-alignment, 24

Lemma
    Factor, 12
    Lifting, 13
level, 23
Lifting Lemma, 13
listing, 72

mangal, iv, 25, 27
mangal requirement, 28
mangrove, iv, 25
    partial order, 28
master condition, 13, 21
measurable, 4
minimal φ-bound, 86
morass, 24
    κ-universal, 61
    augmented, 60
morass code, 63
morass condition, 26

order, 24
outer model programme, iii

partial order, 1
    $\diamondsuit^*$ Oracle, 83
    GCH, 16
    Global Mangrove, 56
    Global Universal Morass, 68
    Mangrove, 28
    Universal Morass, 65
pcf theory, iv
predense $\leq p$ partition, 9
pretame, 6
    below κ, 6
punch line, 59

realm, 43
reduction, 47

simplified morasses, 23
strong, η-, 4
supercompact, λ-, 5
superstrong, 2
    n-, 2

tame, 9
    below κ, 9
top level, 27
tree
    λ-, 74
    λ-Kurepa, 74

universal morass, 61
universal morass condition, 65
Universal Morass Partial Order, 65

ZFC$^-$, 1